

\mathfrak{g} -QUASI-FROBENIUS LIE ALGEBRAS

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ABSTRACT. A Lie version of Turaev's \overline{G} -Frobenius algebras from 2-dimensional homotopy quantum field theory is proposed. The foundation for this Lie version is a structure we call a \mathfrak{g} -quasi-Frobenius Lie algebra for \mathfrak{g} a finite dimensional Lie algebra. The latter consists of a quasi-Frobenius Lie algebra (\mathfrak{q}, β) together with a left \mathfrak{g} -module structure which acts on \mathfrak{q} via derivations and for which β is \mathfrak{g} -invariant. Geometrically, \mathfrak{g} -quasi-Frobenius Lie algebras are the Lie algebra structures associated to symplectic Lie groups with an action by a Lie group G which acts via symplectic Lie group automorphisms. In addition to geometry, \mathfrak{g} -quasi-Frobenius Lie algebras can also be motivated from the point of view of category theory. Specifically, \mathfrak{g} -quasi-Frobenius Lie algebras correspond to *quasi Frobenius Lie objects* in $\mathbf{Rep}(\mathfrak{g})$. If \mathfrak{g} is now equipped with a Lie bialgebra structure, then the categorical formulation of \overline{G} -Frobenius algebras given in [18] suggests that the Lie version of a \overline{G} -Frobenius algebra is a quasi-Frobenius Lie object in $\mathbf{Rep}(D(\mathfrak{g}))$, where $D(\mathfrak{g})$ is the associated (semiclassical) Drinfeld double. We show that if \mathfrak{g} is a quasitriangular Lie bialgebra, then every \mathfrak{g} -quasi-Frobenius Lie algebra has an induced $D(\mathfrak{g})$ -action which gives it the structure of a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra.

1. INTRODUCTION

Renewed interest in Frobenius algebras arose shortly after Witten's introduction of *Topological Quantum Field Theory* (TQFT) in [27]. Shortly afterwards, Atiyah proposed a set of axioms for TQFT [3], thus making Witten's work more accessible to the mathematical community. Working from Atiyah's axioms, L. Abrams showed that 2-dimensional TQFTs are classified by commutative Frobenius algebras [1]. Hence, in the 2-dimensional case, the algebraic structure of a TQFT is that of a Frobenius algebra.

The notion of a $(d+1)$ -dimensional TQFT was generalized to a $(d+1)$ -dimensional *Homotopy Quantum Field Theory* (HQFT) by V. Turaev in [25] by equipping closed d -manifolds and $(d+1)$ -dimensional cobordisms with homotopy classes of maps into a target space X . In the special case when X is a $K(\overline{G}, 1)$ -space for \overline{G} a finite group, one finds that the 2-dimensional

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HQFTs are classified by Frobenius algebras with a \overline{G} -grading and a \overline{G} -action which satisfies a number of conditions [25, 17]. These Frobenius algebras came to be called \overline{G} -Frobenius algebras (or *crossed \overline{G} -algebras*).

In [18], a categorical formulation of \overline{G} -Frobenius algebras was presented where \overline{G} -Frobenius algebras were shown to correspond to certain types of Frobenius objects in $\mathbf{Rep}(D(k[\overline{G}]))$, the braided monoidal category of finite dimensional left $D(k[\overline{G}])$ -modules, where $D(k[\overline{G}])$ is the Drinfeld double of the group ring $k[\overline{G}]$ with its usual Hopf structure. Now the semiclassical analogue of $D(k[\overline{G}])$ (or more generally $D(H)$ for H a finite dimensional Hopf algebra) is $D(\mathfrak{g})$, the Drinfeld double of a finite dimensional Lie bialgebra (\mathfrak{g}, γ) [9, 19, 7, 10]. The relationship between \overline{G} -Frobenius algebras and $D(k[\overline{G}])$ in [18] motivates the following question:

With (\mathfrak{g}, γ) fixed, what structure plays the role of a \overline{G} -Frobenius algebra for $D(\mathfrak{g})$?

Since $D(\mathfrak{g})$ is the Lie version of $D(k[\overline{G}])$, the structure in question should be the Lie version of a \overline{G} -Frobenius algebra. To answer the aforementioned question, we introduce the notion of \mathfrak{g} -quasi-Frobenius Lie algebras for \mathfrak{g} a finite dimensional Lie algebra. A \mathfrak{g} -quasi-Frobenius Lie algebra consists of a quasi-Frobenius Lie algebra (\mathfrak{q}, β) together with a left \mathfrak{g} -module structure which acts on \mathfrak{q} via derivations and for which β is \mathfrak{g} -invariant. Geometrically, \mathfrak{g} -quasi-Frobenius Lie algebras are the Lie algebra structures of symplectic Lie groups with an action by a Lie group G which acts via symplectic Lie group automorphisms. We call the aforementioned structures G -symplectic Lie groups.

Interestingly, \mathfrak{g} -quasi-Frobenius Lie algebras have a categorical formulation. To obtain this formulation, we introduce the notion of a *quasi-Frobenius Lie object* for any additive symmetric monoidal category. The work of Goyvaerts and Vercuysse on the categorification of Lie algebras [12] provides the foundation for defining quasi-Frobenius Lie objects. The latter then yields an alternate (yet equivalent) definition of a \mathfrak{g} -quasi-Frobenius Lie algebra: *a \mathfrak{g} -quasi Frobenius Lie algebra is simply a quasi Frobenius Lie object in $\mathbf{Rep}(\mathfrak{g})$* , where $\mathbf{Rep}(\mathfrak{g})$ is the category of finite dimensional representations of \mathfrak{g} . Using the categorical formulation of [18] as motivation, we obtain the Lie version of a \overline{G} -Frobenius algebra: for a fixed finite dimensional Lie bialgebra (\mathfrak{g}, γ) , the Lie version of a \overline{G} -Frobenius algebra is a quasi-Frobenius Lie object in $\mathbf{Rep}(D(\mathfrak{g}))$. In other words, with respect to (\mathfrak{g}, γ) , a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra is the Lie version of a \overline{G} -Frobenius algebra. The definition of $D(\mathfrak{g})$ implies that a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra is equivalent to a quasi-Frobenius Lie algebra (\mathfrak{q}, β) which is both a \mathfrak{g} and \mathfrak{g}^* -quasi-Frobenius Lie algebra where the \mathfrak{g} and \mathfrak{g}^* actions satisfy a certain compatibility condition.

The rest of the paper is organized as follows. In section 2, we give a brief review of quasi-Frobenius Lie algebras, symplectic Lie groups, Lie bialgebras, and the Drinfeld double. In section 3, we formally define \mathfrak{g} -quasi-Frobenius Lie algebras and prove a general result for their construction. We conclude the section with the categorical formulation of these structures. In section 4, G -symplectic Lie groups are introduced. We show that \mathfrak{g} -quasi-Frobenius Lie algebras are the Lie algebra structures of G -symplectic Lie groups. In addition, we show that the category of finite dimensional \mathfrak{g} -quasi-Frobenius Lie algebras is equivalent to the category of simply connected G -symplectic Lie groups where G is also simply connected. In section 5, we focus our attention on $D(\mathfrak{g})$ -quasi-Frobenius Lie algebras. We show that if \mathfrak{g} is a quasi-triangular Lie bialgebra, then every \mathfrak{g} -quasi-Frobenius Lie algebra has an induced $D(\mathfrak{g})$ -action which extends the original \mathfrak{g} -action and gives the underlying quasi-Frobenius Lie algebra the structure of a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra. In particular, for any finite dimensional Lie algebra \mathfrak{g} (viewed as a Lie bialgebra with co-bracket $\gamma \equiv 0$), every \mathfrak{g} -quasi-Frobenius Lie algebra is a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra, where $D(\mathfrak{g})$ is the Drinfeld double of $(\mathfrak{g}, 0)$.

2. PRELIMINARIES

In this section, we briefly review some of the relevant background for the current paper. Throughout this section, k is a field of characteristic zero.

2.1. Quasi-Frobenius Lie Algebras. The definition of a *Frobenius Lie algebra* [22, 23] is modeled after the definition of a Frobenius algebra. Formally, a Frobenius Lie algebra is defined as follows:

Definition 2.1. A *Frobenius Lie algebra* over k is a pair (\mathfrak{g}, α) where \mathfrak{g} is a Lie algebra and $\alpha : \mathfrak{g} \rightarrow k$ is a linear map with the property that the skew-symmetric bilinear form β on \mathfrak{g} defined by

$$\beta(x, y) := \alpha([x, y]) \quad \forall x, y \in \mathfrak{g}$$

is nondegenerate.

As a consequence of the Jacobi identity, the skew-symmetric bilinear form β in Definition 2.1 satisfies the following identity:

$$\beta([x, y], z) + \beta([y, z], x) + \beta([z, x], y) = 0, \quad \forall x, y, z \in \mathfrak{g}. \quad (2.1)$$

Equation (2.1) is equivalent to the statement that β is a 2-cocycle in the Lie algebra cohomology of \mathfrak{g} with values in k (where \mathfrak{g} acts trivially on k). This motivates the following generalization of Definition 2.1:

Definition 2.2. A *quasi-Frobenius Lie algebra* over k is a pair (\mathfrak{g}, β) where \mathfrak{g} is a Lie algebra over k and β is a nondegenerate 2-cocycle in the Lie algebra cohomology of \mathfrak{g} with values in k (where \mathfrak{g} acts trivially on k).

Remark 2.3. A quasi-Frobenius Lie algebra (\mathfrak{g}, β) is a Frobenius Lie algebra iff β is exact, i.e., $\beta(x, y) = (-\delta\alpha)(x, y) := \alpha([x, y])$ for some linear map $\alpha : \mathfrak{g} \rightarrow k$.

Proposition 2.4. *Every 2-dimensional non-abelian Lie algebra admits the structure of a Frobenius Lie algebra. In particular, every 2-dimensional non-abelian quasi-Frobenius Lie algebra is Frobenius.*

Proof. Let \mathfrak{g} be a 2-dimensional non-abelian Lie algebra. Then \mathfrak{g} admits a basis u_1, u_2 such that $[u_1, u_2] = u_2$. Let $\alpha : \mathfrak{g} \rightarrow k$ be the linear map defined by $\alpha(u_1) = 0$ and $\alpha(u_2) = 1$. Then (\mathfrak{g}, α) is a Frobenius Lie algebra.

If (\mathfrak{g}, β) is a quasi-Frobenius Lie algebra, set $\alpha(u_1) = 0$ and $\alpha(u_2) = \beta(u_1, u_2)$. Then it's easy to see that $\beta(x, y) = \alpha([x, y])$ for all $x, y \in \mathfrak{g}$. Hence, (\mathfrak{g}, β) is Frobenius. \square

Remark 2.5. Since every finite dimensional quasi-Frobenius Lie algebra (\mathfrak{g}, β) is also a symplectic vector space, it follows that the dimension of \mathfrak{g} is necessarily even.

Proposition 2.6. *Let \mathfrak{g} be a Lie algebra of dimension n over k and let e_1, e_2, \dots, e_n be a basis of \mathfrak{g} . Then the following statements are equivalent:*

- (1) *There exists $\alpha \in \mathfrak{g}^*$ such that (\mathfrak{g}, α) is a Frobenius Lie algebra.*
- (2) *There exists $\alpha \in \mathfrak{g}^*$ such that $\det(\alpha([e_i, e_j])) \neq 0$*
- (3) *$\det([e_i, e_j]) \neq 0$, where $[e_i, e_j] \in \mathfrak{g}$ are viewed as elements of the symmetric algebra $S(\mathfrak{g})$.*

Proof. (1) \Leftrightarrow (2). Immediate.

(2) \Rightarrow (3). Recall that $S(\mathfrak{g})$ is naturally isomorphic to the polynomial ring in n -variables where the variables are taken to be the basis e_1, e_2, \dots, e_n . Extend the linear map $\alpha : \mathfrak{g} \rightarrow k$ to a unit preserving algebra map $\alpha : S(\mathfrak{g}) \rightarrow k$ via

$$\alpha(v_1 v_2 \cdots v_r) := \alpha(v_1) \alpha(v_2) \cdots \alpha(v_r)$$

for $v_1, \dots, v_r \in \mathfrak{g}$. Then

$$\alpha(\det([e_i, e_j])) = \det(\alpha([e_i, e_j])) \neq 0,$$

which implies that $\det([e_i, e_j]) \neq 0$.

(2) \Leftarrow (3). Let $p = \det([e_i, e_j]) \in S(\mathfrak{g})$. Since $p = p(e_1, \dots, e_n) \neq 0$ and k is infinite, there exists $\lambda_i \in k$ such that $p(\lambda_1, \dots, \lambda_n) \neq 0$ (see Theorem 3.76 of [28]). Let $\alpha : \mathfrak{g} \rightarrow k$ be the linear map defined by $\alpha(e_i) = \lambda_i$ for $i = 1, \dots, n$. As before, extend $\alpha : \mathfrak{g} \rightarrow k$ to an algebra map $\alpha : S(\mathfrak{g}) \rightarrow k$. Then

$$\begin{aligned} \det(\alpha([e_i, e_j])) &= \alpha(\det([e_i, e_j])) \\ &= \alpha(p(e_1, \dots, e_n)) \\ &= p(\alpha(e_1), \dots, \alpha(e_n)) \\ &= p(\lambda_1, \dots, \lambda_n) \\ &\neq 0. \end{aligned}$$

□

We now recall two examples. The first is Frobenius and the second is quasi-Frobenius but not Frobenius [23, 6].

Example 2.7. Let \mathfrak{g} be the 4-dimensional Lie algebra with basis $\{x_1, \dots, x_4\}$ and non-zero commutator relations:

$$[x_1, x_2] = \frac{1}{2}x_2 + x_3, \quad [x_1, x_3] = \frac{1}{2}x_3, \quad [x_1, x_4] = x_4, \quad [x_2, x_3] = x_4$$

Then $\det([x_i, x_j]) = (x_4)^4 \neq 0$, where $[x_i, x_j]$ are regarded as elements of the symmetric algebra $S(\mathfrak{g})$. By Proposition 2.6, there exists a linear map $\alpha : \mathfrak{g} \rightarrow k$ for which (\mathfrak{g}, α) is a Frobenius Lie algebra.

Example 2.8. Let \mathfrak{q} be the 4-dimensional Lie algebra with basis $\{x_1, \dots, x_4\}$ and non-zero commutator relations:

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = x_4$$

Since $\det([x_i, x_j]) = 0$, \mathfrak{q} cannot be Frobenius by Proposition 2.6. However, it does admit the structure of a quasi-Frobenius Lie algebra. As an example of this, let β be the nondegenerate, skew-symmetric bilinear form given by

$$\beta = x_1^* \wedge x_4^* + x_2^* \wedge x_3^*$$

where $\{x_1^*, \dots, x_4^*\}$ is the dual basis. A direct calculation shows that β satisfies the 2-cocycle condition. Hence, (\mathfrak{q}, β) is quasi-Frobenius.

Definition 2.9. Let $(\mathfrak{g}_1, \beta_1)$ and $(\mathfrak{g}_2, \beta_2)$ be quasi-Frobenius Lie algebras. A *quasi-Frobenius Lie algebra homomorphism* from $(\mathfrak{g}_1, \beta_1)$ to $(\mathfrak{g}_2, \beta_2)$ is a Lie algebra homomorphism $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that $\varphi^* \beta_2 = \beta_1$, that is,

$$\beta_1(u, v) = \beta_2(\varphi(u), \varphi(v)), \quad \forall u, v \in \mathfrak{g}_1. \quad (2.2)$$

If $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ satisfies (2.2) and is also a Lie algebra isomorphism, then φ is an isomorphism of quasi-Frobenius Lie algebras.

Proposition 2.10. Let $\varphi : (\mathfrak{g}_1, \beta_1) \rightarrow (\mathfrak{g}_2, \beta_2)$ be a quasi-Frobenius Lie algebra map. If $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_2 < \infty$, then φ is an isomorphism of quasi-Frobenius Lie algebras.

Proof. Since $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_2 < \infty$, it suffices to show that φ is injective. Let $u \in \mathfrak{g}_1$ be any nonzero element. Since β is nondegenerate, there exists $v \in \mathfrak{g}_1$ such that $\beta(u, v) \neq 0$. Hence,

$$\beta_2(\varphi(u), \varphi(v)) = \beta_1(u, v) \neq 0,$$

which implies that $\varphi(u) \neq 0$. This completes the proof. □

2.2. Symplectic Lie Groups. In this section, we recall the correspondence between *symplectic Lie groups* [4, 8] and quasi-Frobenius Lie algebras.

Definition 2.11. A *symplectic Lie group* is a pair (G, ω) where G is a Lie group and ω is a left-invariant symplectic form on G .

The next result shows that the Lie algebra of a symplectic Lie group is naturally a quasi-Frobenius Lie algebra.

Proposition 2.12. *Let (G, ω) be a symplectic Lie group. Then (\mathfrak{g}, ω_e) is a quasi-Frobenius Lie algebra.*

Proof. Let $\mathfrak{X}_l(G)$ denote the space of left-invariant vector fields on G and endow $\mathfrak{g} := T_e G$ with the Lie algebra structure of $\mathfrak{X}_l(G)$. Also, let \tilde{x} denote the left-invariant vector field associated with $x \in \mathfrak{g}$. We now show that (\mathfrak{g}, ω_e) is a quasi-Frobenius Lie algebra. Since $\omega_g|_{T_g G}$ is nondegenerate for all $g \in G$ (in particular for $g = e$), it only remains to show that ω_e is a 2-cocycle of \mathfrak{g} with values in \mathbb{R} (where \mathfrak{g} acts trivially on \mathbb{R}).

First, note that for any $x, y \in \mathfrak{g}$, $\omega(\tilde{x}, \tilde{y})$ is a constant function on G . Indeed, for $g \in G$

$$\begin{aligned} (\omega(\tilde{x}, \tilde{y}))(g) &:= \omega_g(\tilde{x}_g, \tilde{y}_g) \\ &= \omega_g((l_g)_* x, (l_g)_* y) \\ &= (l_g^* \omega)_e(x, y) \\ &= \omega_e(x, y) \end{aligned}$$

where the last equality follows from the fact that ω is left-invariant. This fact along with the fact the ω is closed implies that $\omega_e \in Z^2(\mathfrak{g}; \mathbb{R})$:

$$\begin{aligned} 0 &= d\omega(\tilde{x}, \tilde{y}, \tilde{z}) \\ &= \tilde{x}(\omega(\tilde{y}, \tilde{z})) - \tilde{y}(\omega(\tilde{x}, \tilde{z})) + \tilde{z}(\omega(\tilde{x}, \tilde{y})) \\ &\quad - \omega([\tilde{x}, \tilde{y}], \tilde{z}) + \omega([\tilde{x}, \tilde{z}], \tilde{y}) - \omega([\tilde{y}, \tilde{z}], \tilde{x}) \\ &= -\omega([\tilde{x}, \tilde{y}], \tilde{z}) - \omega([\tilde{z}, \tilde{x}], \tilde{y}) - \omega([\tilde{y}, \tilde{z}], \tilde{x}). \end{aligned}$$

Evaluating the last equality at $e \in G$ and multiplying by -1 gives the 2-cocycle condition on ω_e :

$$\omega_e([x, y], z) + \omega_e([z, x], y) + \omega_e([y, z], x) = 0.$$

Hence, (\mathfrak{g}, ω_e) is a quasi-Frobenius Lie algebra. \square

Proposition 2.13. *Let G be a Lie group whose Lie algebra \mathfrak{g} carries the structure of a quasi-Frobenius Lie algebra with 2-cocycle β . Define $\tilde{\beta} \in \Omega^2(G)$ by*

$$\tilde{\beta}_g := (l_{g^{-1}})^* \beta \in \wedge^2 T_g^* G, \quad \forall g \in G$$

where $l_g : G \rightarrow G$ is left translation by g . Then $(G, \tilde{\beta})$ is a symplectic Lie group whose associated quasi-Frobenius Lie algebra is $(\mathfrak{g}, \tilde{\beta}_e) = (\mathfrak{g}, \beta)$.

Proof. It follows immediately from the definition that $\tilde{\beta}$ is left-invariant, that is, $(l_g)^*\tilde{\beta} = \tilde{\beta}$ for all $g \in G$. Moreover, since β is nondegenerate, $\tilde{\beta}$ must be nondegenerate as well. To see that $d\tilde{\beta} = 0$, it suffices to show that $d\tilde{\beta}(\tilde{x}, \tilde{y}, \tilde{z}) = 0$ for all left-invariant vector fields \tilde{x} , \tilde{y} , and \tilde{z} . Since $\tilde{\beta}$ is left-invariant, it follows that $\tilde{\beta}(\tilde{x}, \tilde{y}) = \tilde{\beta}_e(x, y) = \beta(x, y)$ is a constant function on G for all left-invariant vector fields \tilde{x} and \tilde{y} , where $\tilde{x}_e = x$ and $\tilde{y}_e = y$. In particular,

$$\tilde{\beta}([\tilde{x}, \tilde{y}], \tilde{z}) = \beta([x, y], z).$$

The proof of Proposition 2.12 shows that if $\tilde{\beta}$ is left-invariant, we have

$$\begin{aligned} d\tilde{\beta}(\tilde{x}, \tilde{y}, \tilde{z}) &= -\tilde{\beta}([\tilde{x}, \tilde{y}], \tilde{z}) - \tilde{\beta}([\tilde{z}, \tilde{x}], \tilde{y}) - \tilde{\beta}([\tilde{y}, \tilde{z}], \tilde{x}) \\ &= -\beta([x, y], z) - \beta([z, x], y) - \beta([y, z], x). \end{aligned}$$

Since $\beta \in Z^2(\mathfrak{g}; \mathbb{R})$, the last equality must be zero. Hence, $(G, \tilde{\beta})$ is a symplectic Lie group. \square

Definition 2.14. Let (G, ω) and (H, σ) be symplectic Lie groups. A homomorphism of symplectic Lie groups is a Lie group homomorphism $\varphi : G \rightarrow H$ such that $\varphi^*\sigma = \omega$.

Lemma 2.15. Let (G, ω) and (H, σ) be symplectic Lie groups and let $\varphi : G \rightarrow H$ be a Lie group homomorphism. Then $\varphi^*\sigma = \omega$ iff $(\varphi^*\sigma)_e = \omega_e$.

Proof. (\Rightarrow) Suppose $(\varphi^*\sigma) = \omega$. By definition, $(\varphi^*\sigma)_g = \omega_g$ for all $g \in G$. In particular, the equality holds for $g = e$.

(\Leftarrow) Now suppose $(\varphi^*\sigma)_e = \omega_e$. Let $g \in G$ and $x, y \in T_g G$. Then

$$\begin{aligned} (\varphi^*\sigma)_g(x, y) &= \sigma_{\varphi(g)}(\varphi_{*,g}(x), \varphi_{*,g}(y)) \\ &= [(l_{\varphi(g^{-1})})^*\sigma_e](\varphi_{*,g}(x), \varphi_{*,g}(y)) \\ &= \sigma_e((l_{\varphi(g^{-1})} \circ \varphi)_{*,g}(x), (l_{\varphi(g^{-1})} \circ \varphi)_{*,g}(y)) \\ &= \sigma_e((\varphi \circ l_{g^{-1}})_{*,g}(x), (\varphi \circ l_{g^{-1}})_{*,g}(y)) \\ &= (\varphi^*\sigma)_e((l_{g^{-1}})_{*,g}(x), (l_{g^{-1}})_{*,g}(y)) \\ &= \omega_e((l_{g^{-1}})_{*,g}(x), (l_{g^{-1}})_{*,g}(y)) \\ &= [(l_{g^{-1}})^*\omega_e](x, y) \\ &= \omega_g(x, y), \end{aligned}$$

where the second and last equalities follow from the left-invariance of σ and ω respectively and the fourth equality follows from the fact that φ is a group homomorphism. This completes the proof. \square

Proposition 2.16. Let $\varphi : (G, \omega) \rightarrow (H, \sigma)$ be a homomorphism of symplectic Lie groups. Then

$$\varphi_{*,e} : (\mathfrak{g}, \omega_e) \rightarrow (\mathfrak{h}, \sigma_e)$$

is a homomorphism of quasi-Frobenius Lie algebras.

Proof. This follows immediately from the properties of φ . \square

Proposition 2.17. *Let $\psi : (\mathfrak{g}, \beta) \rightarrow (\mathfrak{h}, \sigma)$ be a homomorphism of quasi-Frobenius Lie algebras. Let G be the simply connected Lie group whose Lie algebra is \mathfrak{g} and let H be any Lie group whose Lie algebra is \mathfrak{h} . Let $(G, \tilde{\beta})$ and $(H, \tilde{\sigma})$ be the symplectic Lie groups associated to (\mathfrak{g}, β) and (\mathfrak{h}, σ) respectively (see Proposition 2.13). Then there exists a unique symplectic Lie group homomorphism*

$$\hat{\psi} : (G, \tilde{\beta}) \rightarrow (H, \tilde{\sigma})$$

such that $\hat{\psi}_{*,e} = \psi$.

Proof. Since G is simply connected, there exists a unique Lie group homomorphism $\hat{\psi} : G \rightarrow H$ such that $\hat{\psi}_{*,e} = \psi$. It only remains to show that $\hat{\psi}^* \tilde{\sigma} = \tilde{\beta}$. By Lemma 2.15, it suffices to show that $(\hat{\psi}^* \tilde{\sigma})_e = \tilde{\beta}_e = \beta$. To do this, let $x, y \in \mathfrak{g}$. Then

$$\begin{aligned} (\hat{\psi}^* \tilde{\sigma})_e(x, y) &= \tilde{\sigma}_{\hat{\psi}(e)}(\hat{\psi}_{*,e}(x), \hat{\psi}_{*,e}(y)) \\ &= \tilde{\sigma}_e(\psi(x), \psi(y)) \\ &= \sigma(\psi(x), \psi(y)) \\ &= (\psi^* \sigma)(x, y) \\ &= \beta(x, y). \end{aligned}$$

This completes the proof. \square

Theorem 2.18. *Let **SCSLG** be the category of simply connected symplectic Lie groups and let **qFLA** be the category of finite dimensional quasi-Frobenius Lie algebras. Let F be the functor from **SCSLG** to **qFLA** which sends (G, ω) to (\mathfrak{g}, ω_e) and $\varphi : (G, \omega) \rightarrow (H, \sigma)$ to $\varphi_{*,e} : (\mathfrak{g}, \omega_e) \rightarrow (\mathfrak{h}, \sigma_e)$. Then F is an equivalence of categories.*

Proof. Theorem 2.18 follows from the well known correspondence between simply connected Lie groups and finite dimensional Lie algebras combined with Proposition 2.12, Proposition 2.13, Proposition 2.16, and Proposition 2.17. \square

As an example, we now recall the symplectic Lie group structure on the affine Lie group $A(n, \mathbb{R})$ (c.f., [2, 21, 23]).

Example 2.19. Recall that $A(n, \mathbb{R})$ is the Lie group consisting of $(n+1) \times (n+1)$ matrices of the form

$$A(n, \mathbb{R}) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in GL(n, \mathbb{R}), v \in \mathbb{R}^n \right\}.$$

The associated Lie algebra is then

$$\mathfrak{a}(n, \mathbb{R}) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{gl}(n, \mathbb{R}), v \in \mathbb{R}^n \right\}.$$

From the definition of $A(n, \mathbb{R})$, we see that $A(n, \mathbb{R})$ is even dimensional with $\dim A(n, \mathbb{R}) = \dim \mathfrak{a}(n, \mathbb{R}) = n^2 + n = n(n+1)$. Let E_{ij} denote the $(n+1) \times$

$(n+1)$ matrix with 1 in the (i, j) -component and all other components zero. Then $\{E_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq n+1}$ is a basis on $\mathfrak{a}(n, \mathbb{R})$. Let $\{E_{ij}^*\}_{1 \leq i \leq n, 1 \leq j \leq n+1}$ denote the corresponding dual basis. Define

$$\alpha = E_{12}^* + E_{23}^* + \cdots + E_{n,n+1}^*$$

and $\beta(X, Y) := -\delta\alpha(X, Y) = \alpha([X, Y])$ for all $X, Y \in \mathfrak{a}(n, \mathbb{R})$. Since

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj},$$

we see that

$$\beta(E_{ij}, E_{kl}) = \delta_{jk}\delta_{l,i+1} - \delta_{li}\delta_{j,k+1}. \quad (2.3)$$

Careful consideration of (2.3) shows that $\beta := -\delta\alpha \in Z^2(\mathfrak{a}(n, \mathbb{R}); \mathbb{R})$ is non-degenerate. Hence, $(\mathfrak{a}(n, \mathbb{R}), \alpha)$ is a Frobenius Lie algebra. (In particular, $(\mathfrak{a}(n, \mathbb{R}), \beta)$ is a quasi-Frobenius Lie algebra.) Let $\tilde{\beta} \in \Omega^2(A(n, \mathbb{R}))$ be the left-invariant 2-form on $A(n, \mathbb{R})$ associated to β . Then $(A(n, \mathbb{R}), \tilde{\beta})$ is a symplectic Lie group. Furthermore, since $\beta := -\delta\alpha$, it follows that $\tilde{\beta}$ is exact. Specifically,

$$\tilde{\beta} = -d\tilde{\alpha}$$

where $\tilde{\alpha} \in \Omega^1(A(n, \mathbb{R}))$ is the left-invariant 1-form on $A(n, \mathbb{R})$ associated to α .

2.3. Lie bialgebras & the Drinfeld Double.

Definition 2.20. A *Lie bialgebra* over a field k is a pair (\mathfrak{g}, γ) where \mathfrak{g} is a Lie algebra over k and $\gamma : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ is a skew-symmetric linear map such that

1. $\gamma^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie bracket on \mathfrak{g}^* , where the dual map γ^* is restricted to $\mathfrak{g}^* \otimes \mathfrak{g}^* \subset (\mathfrak{g} \otimes \mathfrak{g})^*$;
2. γ is a 1-cocycle on \mathfrak{g} with values in $\mathfrak{g} \otimes \mathfrak{g}$, where \mathfrak{g} acts on $\mathfrak{g} \otimes \mathfrak{g}$ via the adjoint action.

γ is called the *cobracket* or *co-commutator*.

Condition 2 in Definition 2.20 is equivalent to the condition

$$\gamma([x, y]) = ad_x^{(2)}\gamma(y) - ad_y^{(2)}\gamma(x), \quad \forall x, y \in \mathfrak{g}$$

where the linear map $ad_x^{(2)} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is the adjoint action of $x \in \mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$. Explicitly, $ad_x^{(2)}$ is defined via

$$ad_x^{(2)}(y \otimes z) = ad_x(y) \otimes z + y \otimes ad_x(z) = [x, y] \otimes z + y \otimes [x, z]$$

for $y, z \in \mathfrak{g}$.

Definition 2.21. Let $(\mathfrak{g}, \gamma_{\mathfrak{g}})$ and $(\mathfrak{h}, \gamma_{\mathfrak{h}})$ be Lie bialgebras. A *Lie bialgebra homomorphism* from $(\mathfrak{g}, \gamma_{\mathfrak{g}})$ to $(\mathfrak{h}, \gamma_{\mathfrak{h}})$ is a Lie algebra map $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ such that

$$(\varphi \otimes \varphi) \circ \gamma_{\mathfrak{g}} = \gamma_{\mathfrak{h}} \circ \varphi.$$

Example 2.22. Any Lie algebra \mathfrak{g} can be turned into a Lie bialgebra by taking the cobracket $\gamma \equiv 0$. $(\mathfrak{g}, 0)$ is the *trivial* Lie bialgebra structure on \mathfrak{g} .

The next result shows that the notion of a Lie bialgebra is self-dual for the finite dimensional case.

Proposition 2.23. *Let $(\mathfrak{g}, \gamma_{\mathfrak{g}})$ be a finite dimensional Lie bialgebra and let $\gamma_{\mathfrak{g}^*} : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$ be the dual of the Lie bracket on \mathfrak{g} . Then $(\mathfrak{g}^*, \gamma_{\mathfrak{g}^*})$ is a Lie bialgebra, where the Lie bracket on \mathfrak{g}^* is given by the dual of $\gamma_{\mathfrak{g}}$.*

For a Lie algebra \mathfrak{g} , the simplest way to obtain an element of $Z_{ad}^1(\mathfrak{g}; \mathfrak{g} \otimes \mathfrak{g})$ is to turn to the 0-cochains and take their coboundaries. This raises the following natural question: given $r \in \mathfrak{g} \otimes \mathfrak{g}$, when does $\delta r \in Z_{ad}^1(\mathfrak{g}; \mathfrak{g} \otimes \mathfrak{g})$ define a Lie bialgebra structure on \mathfrak{g} ? To answer this question, let

$$r = \sum_i a_i \otimes b_i,$$

and define

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \quad (2.4)$$

where

$$[r_{12}, r_{13}] := \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j \quad (2.5)$$

$$[r_{12}, r_{23}] := \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j \quad (2.6)$$

$$[r_{13}, r_{23}] := \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j]. \quad (2.7)$$

Definition 2.24. A *coboundary Lie bialgebra* is a Lie bialgebra (\mathfrak{g}, γ) such that $\gamma = \delta r$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$. The element r is called the *r-matrix*.

The next result provides a necessary and sufficient condition for an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ to define a Lie bialgebra structure on \mathfrak{g} .

Proposition 2.25. *Let \mathfrak{g} be a Lie algebra. Then $(\mathfrak{g}, \delta r)$ is a Lie bialgebra iff*

- (i) $r + \sigma(r)$ is invariant under the adjoint action of \mathfrak{g} on $\mathfrak{g} \otimes \mathfrak{g}$, where $\sigma : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is the unique linear map defined by $x \otimes y \mapsto y \otimes x$ for $x, y \in \mathfrak{g}$;
- (ii) $[[r, r]]$ is invariant under the adjoint action of \mathfrak{g} on $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

Proof. See pp. 51-54 of [7]. □

The simplest way to ensure that condition (ii) of Proposition 2.25 is satisfied is to demand that

$$[[r, r]] = 0. \quad (2.8)$$

Equation 2.8 is called the *classical Yang-Baxter equation* (CYBE). The CYBE motivates the following definition:

Definition 2.26. A coboundary Lie bialgebra $(\mathfrak{g}, \delta r)$ is *quasitriangular* if r is a solution of the CYBE. Furthermore, if r is skew-symmetric, that is, $r \in \mathfrak{g} \wedge \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$, then $(\mathfrak{g}, \delta r)$ is said to be *triangular*.

Example 2.27. Let \mathfrak{g} be the two dimensional Lie algebra with basis x, y and commutator relation $[x, y] = x$. Define $r = y \wedge x$. Then $(\mathfrak{g}, \delta r)$ is a triangular Lie bialgebra, where $\gamma := \delta r$ is given explicitly by

$$\gamma(x) = 0, \quad \gamma(y) = x \wedge y.$$

Before turning to the Drinfeld double, we recall the following notion:

Definition 2.28. Let \mathfrak{g} be a Lie algebra and let $\langle \cdot, \cdot \rangle$ be a bilinear form on \mathfrak{g} . \mathfrak{g} is *ad-invariant* with respect to $\langle \cdot, \cdot \rangle$ if

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle, \quad \forall x, y, z \in \mathfrak{g}. \quad (2.9)$$

Now let $(\mathfrak{g}, \gamma_{\mathfrak{g}})$ be a finite dimensional Lie bialgebra and let $(\mathfrak{g}^*, \gamma_{\mathfrak{g}^*})$ be the associated dual Lie bialgebra. Consider the direct sum

$$\mathfrak{g} \oplus \mathfrak{g}^*$$

and equip it with the symmetric, nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle x + \xi, y + \eta \rangle = \xi(y) + \eta(x),$$

where we write $x + \xi$ and $y + \eta$ for $(x, \xi), (y, \eta) \in \mathfrak{g} \oplus \mathfrak{g}^*$. The Drinfeld double of $(\mathfrak{g}, \gamma_{\mathfrak{g}})$, denoted by $D(\mathfrak{g})$, is the unique quasitriangular Lie bialgebra which satisfies the following conditions:

- (1) As a vector space,

$$D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*.$$

- (2) As a Lie algebra, $D(\mathfrak{g})$ is ad-invariant with respect to the inner product $\langle \cdot, \cdot \rangle$ and contains \mathfrak{g} and \mathfrak{g}^* as Lie subalgebras.

- (3) The cobracket on $D(\mathfrak{g})$ is defined by $\gamma_D := \gamma_{\mathfrak{g}} - \gamma_{\mathfrak{g}^*}$.

Let $[\cdot, \cdot]_D$, $[\cdot, \cdot]_{\mathfrak{g}}$, and $[\cdot, \cdot]_{\mathfrak{g}^*}$ denote the Lie brackets on $D(\mathfrak{g})$, \mathfrak{g} , and \mathfrak{g}^* respectively. Condition (2) implies that

$$[x, y]_D = [x, y]_{\mathfrak{g}}, \quad [\xi, \eta]_D = [\xi, \eta]_{\mathfrak{g}^*}, \quad [x, \xi]_D = ad_x^* \xi - ad_{\xi}^* x$$

for all $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$, where ad^* denotes the coadjoint action of \mathfrak{g} on \mathfrak{g}^* and \mathfrak{g}^* on \mathfrak{g} . Explicitly, $ad_x^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ and $ad_{\xi}^* : \mathfrak{g} \rightarrow \mathfrak{g}$ are defined by $ad_x^* := -ad_x^t$ and $ad_{\xi}^* := -ad_{\xi}^t$ where ad_x^t and ad_{ξ}^t are the ordinary duals of $ad_x : \mathfrak{g} \rightarrow \mathfrak{g}$ and $ad_{\xi} : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$. In dealing with the Drinfeld double, we will drop the “ D ”, “ \mathfrak{g} ”, and “ \mathfrak{g}^* ” that appear as subscripts in the Lie brackets of $D(\mathfrak{g})$, \mathfrak{g} , and \mathfrak{g}^* respectively. Condition (2) implies that the triple $(D(\mathfrak{g}), \mathfrak{g}, \mathfrak{g}^*)$ is a *Manin triple* with respect to the inner product $\langle \cdot, \cdot \rangle$. In fact, there is a one to one correspondence between finite dimensional Lie bialgebras and Manin triples (see [7]).

Lastly, condition (3) implies that $D(\mathfrak{g})$ is quasitriangular with r-matrix

$$r = \sum_i e_i \otimes e_i^*$$

where e_1, \dots, e_n is any basis on \mathfrak{g} and e_1^*, \dots, e_n^* is the corresponding dual basis.

Example 2.29. Let (\mathfrak{g}, γ) be the 2-dimensional Lie bialgebra with basis x, y satisfying $[x, y] = x$ and cobracket $\gamma(x) = 0$ and $\gamma(y) = x \wedge y$. Let x^*, y^* denote the corresponding dual basis. The commutator relations on $D(\mathfrak{g})$ are

$$\begin{aligned} [x, y] &= x, & [x^*, y^*] &= y^*, & [x, x^*] &= -y^*, & [x, y^*] &= 0 \\ [y, x^*] &= x^* + y, & [y, y^*] &= -x \end{aligned}$$

The r-matrix is $r = x \otimes x^* + y \otimes y^*$.

3. \mathfrak{g} -QUASI-FROBENIUS LIE ALGEBRAS

We begin with the formal definition:

Definition 3.1. A \mathfrak{g} -quasi-Frobenius Lie algebra is a triple $(\mathfrak{q}, \beta, \rho)$ such that (\mathfrak{q}, β) is a quasi-Frobenius Lie algebra and $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{q})$, $x \mapsto \rho_x$ is a left \mathfrak{g} -module structure on \mathfrak{q} such that

- (i) ρ_x is a derivation on \mathfrak{q} for all $x \in \mathfrak{g}$
- (ii) $\beta(\rho_x(u), v) + \beta(u, \rho_x(v)) = 0$ for all $x \in \mathfrak{g}$, $u, v \in \mathfrak{q}$ (\mathfrak{g} -invariance)

In this section, we prove a result for the general construction of \mathfrak{g} -quasi-Frobenius Lie algebras. Before doing so, we make the following observation:

Proposition 3.2. Let (\mathfrak{q}, β) be a quasi-Frobenius Lie algebra and let $\text{Aut}(\mathfrak{q}, \beta)$ be the automorphism group of (\mathfrak{q}, β) . Then $\text{Aut}(\mathfrak{q}, \beta)$ is an embedded Lie subgroup of $GL(\mathfrak{q})$.

Proof. As a set, $\text{Aut}(\mathfrak{q}, \beta) = \text{Aut}(\mathfrak{q}) \cap \text{Sp}(\mathfrak{q}, \beta)$ where $\text{Aut}(\mathfrak{q})$ is the group of automorphisms of the Lie algebra \mathfrak{q} and $\text{Sp}(\mathfrak{q}, \beta)$ is the group of linear symplectomorphisms of (\mathfrak{q}, β) , where the latter is regarded as a symplectic vector space. Since $\text{Aut}(\mathfrak{q})$ and $\text{Sp}(\mathfrak{q}, \beta)$ are both closed subgroups of $GL(\mathfrak{q})$, each being the zero set of a collection of polynomials, $\text{Aut}(\mathfrak{q}, \beta)$ is also a closed subgroup of $GL(\mathfrak{q})$. By the closed subgroup theorem [26], $\text{Aut}(\mathfrak{q}, \beta)$ is an embedded Lie subgroup of $GL(\mathfrak{q})$. \square

Proposition 3.3. Let (\mathfrak{q}, β) be a quasi-Frobenius Lie algebra and let

$$\rho : G \rightarrow \text{Aut}(\mathfrak{q}, \beta) \subset GL(\mathfrak{q}), \quad g \mapsto \rho_g$$

be a Lie group homomorphism. Define

$$\rho' := \rho_{*,e} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{q}), \quad x \mapsto \rho'_x.$$

Then $(\mathfrak{q}, \beta, \rho')$ is a \mathfrak{g} -quasi-Frobenius Lie algebra. In particular, if G is any Lie subgroup of $\text{Aut}(\mathfrak{q}, \beta)$, then (\mathfrak{q}, β) admits the structure of a \mathfrak{g} -quasi-Frobenius Lie algebra.

Proof. Since ρ is a Lie group homomorphism, it immediately follows that $\rho' : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{q})$ is a representation of \mathfrak{g} on \mathfrak{q} . We now show that

$$\rho_x([u, v]) = [\rho_x(u), v] + [u, \rho_x(v)] \tag{3.1}$$

and

$$\beta(\rho_x(u), v) + \beta(u, \rho_x(v)) = 0 \tag{3.2}$$

for all $x \in \mathfrak{g}$ and $u, v \in \mathfrak{q}$. To do this, fix a basis e_1, e_2, \dots, e_n on \mathfrak{q} . Since $\rho_{\exp(tx)}(u), \rho_{\exp(tx)}(v) \in \mathfrak{q}$, we have

$$\rho_{\exp(tx)}(u) = \sum_i a_i(t)e_i, \quad \rho_{\exp(tx)}(v) = \sum_i b_i(t)e_i \quad (3.3)$$

for some smooth functions $a_i(t), b_i(t)$, $i = 1, \dots, n$. Hence,

$$\rho'_x(u) = \sum_i \dot{a}_i(0)e_i, \quad \rho'_x(v) = \sum_i \dot{b}_i(0)e_i. \quad (3.4)$$

Since $\rho_g \in \text{Aut}(\mathfrak{q}, \beta)$ for all $g \in G$, we have

$$\rho_{\exp(tx)}([u, v]) = [\rho_{\exp(tx)}(u), \rho_{\exp(tx)}(v)]. \quad (3.5)$$

Substituting (3.3) into the right side of (3.5) and applying $\frac{d}{dt}|_{t=0}$ to both sides of (3.5) gives

$$\begin{aligned} \rho'_x([u, v]) &= \frac{d}{dt}\bigg|_{t=0} [\rho_{\exp(tx)}(u), \rho_{\exp(tx)}(v)] \\ &= \frac{d}{dt}\bigg|_{t=0} \sum_{i,j} a_i(t)b_j(t)[e_i, e_j] \\ &= \sum_{i,j} (\dot{a}_i(0)b_j(0)[e_i, e_j] + a_i(0)\dot{b}_j(0)[e_i, e_j]) \\ &= [\rho'_x(u), v] + [u, \rho'_x(v)], \end{aligned} \quad (3.6)$$

which proves (3.1).

For equation (3.2), note that

$$\beta(\rho_{\exp(tx)}(u), \rho_{\exp(tx)}(v)) = \beta(u, v) \quad (3.7)$$

since $\rho_g \in \text{Aut}(\mathfrak{q}, \beta)$ for all $g \in G$. Substituting (3.3) into the left side of (3.7) and applying $\frac{d}{dt}|_{t=0}$ to both sides of (3.7) gives

$$\beta(\rho'_x(u), v) + \beta(u, \rho'_x(v)) = 0.$$

This completes the proof. \square

A trivial example of a \mathfrak{g} -quasi-Frobenius Lie algebra is obtained by equipping any quasi-Frobenius Lie algebra with the trivial \mathfrak{g} -action. We now consider a more interesting example which is an application of Proposition 3.3.

Example 3.4. Let \mathfrak{q} be the 4-dimensional Lie algebra $\{e_1, e_2, e_3, e_4\}$ whose non-zero commutator relations are given by [6]:

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = 2e_4, \quad [e_2, e_3] = e_4.$$

Let $\alpha : \mathfrak{q} \rightarrow \mathbb{R}$ be the linear map defined by $\alpha(e_i) = 0$ for $i = 1, 2, 3$ and $\alpha(e_4) = 1$. Define $\beta(u, v) := \alpha([u, v])$ for all $u, v \in \mathfrak{q}$. Then the matrix

representation of β with respect to the basis $\{e_1, e_2, e_3, e_4\}$ is

$$(\beta_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}$$

Hence, β is nondegenerate which shows that (\mathfrak{q}, α) is a Frobenius Lie algebra. Let G be the set of linear isomorphisms on \mathfrak{q} whose matrix representations with respect to $\{e_1, e_2, e_3, e_4\}$ is given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 1/b & 0 \\ a & 0 & 0 & 1 \end{pmatrix} \mid a, c \in \mathbb{R}, b > 0 \right\}. \quad (3.8)$$

A direct calculation shows that G is a 3-dimensional non-abelian, connected Lie subgroup of $Aut(\mathfrak{q}, \beta)$. Let $\rho : G \rightarrow Aut(\mathfrak{q}, \beta) \subset GL(\mathfrak{q})$ be the inclusion map (which is clearly a Lie group homomorphism). Proposition 3.3 implies that $(\mathfrak{q}, \beta, \rho')$ is a \mathfrak{g} -quasi-Frobenius Lie algebra, where $\rho' := \rho_{*,e}$. As a Lie algebra, \mathfrak{g} has basis

$$x_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad x_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x_3 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.9)$$

where we have identified G with its matrix representations in (3.8). The non-zero commutator relations are

$$[x_2, x_3] = 2x_3.$$

Let $a = a_1x_1 + a_2x_2 + a_3x_3 \in \mathfrak{g}$. Since $\rho : G \rightarrow Aut(\mathfrak{q}, \beta) \subset GL(\mathfrak{q})$ is just the inclusion map, it follows that the matrix representation of $\rho'_a : \mathfrak{q} \rightarrow \mathfrak{q}$ with respect to the basis $\{e_1, e_2, e_3, e_4\}$ is simply

$$\rho'_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 \\ 0 & 0 & -a_2 & 0 \\ a_1 & 0 & 0 & 0 \end{pmatrix} \quad (3.10)$$

Since $(\mathfrak{q}, \beta, \rho')$ is a \mathfrak{g} -quasi-Frobenius Lie algebra by Proposition 3.3, ρ'_a acts on \mathfrak{q} via derivations and satisfies

$$\beta(\rho'_a(u), v) + \beta(u, \rho'_a(v)) = 0$$

for all $u, v \in \mathfrak{q}$.

For later use, we conclude this section with the following natural definition:

Definition 3.5. Let $(\mathfrak{q}, \beta, \phi)$ and $(\mathfrak{r}, \sigma, \mu)$ be \mathfrak{g} -quasi-Frobenius Lie algebras. A homomorphism from $(\mathfrak{q}, \beta, \phi)$ to $(\mathfrak{r}, \sigma, \mu)$ is a homomorphism

$$\psi : (\mathfrak{q}, \beta) \rightarrow (\mathfrak{r}, \sigma)$$

of quasi-Frobenius Lie algebras which is also \mathfrak{g} -equivariant, that is,

$$\psi \circ \phi_x = \mu_x \circ \psi$$

for all $x \in \mathfrak{g}$.

3.1. Categorical Formulation. In this section, we apply the idea of categorification to quasi-Frobenius Lie algebras. The upshot of this is the notion of a *quasi-Frobenius Lie object*, which can be viewed as the analogue of a Frobenius object in the current setting. The starting point for this particular step is the categorification of Lie algebra due to Goyvaerts and Vercruysse [12]:

Definition 3.6. A *Lie object* in an additive symmetric monoidal category $(\mathcal{C}, \otimes, I, \Phi, l, r, c)$ is a pair (L, b) where L is an object of \mathcal{C} and $b : L \otimes L \rightarrow L$ is a morphism such that

- (i) $b + b \circ c = 0_{L \otimes L, L}$
- (ii) $b \circ (id_L \otimes b) \circ (id_{L \otimes (L \otimes L)} + c_{L \otimes L, L} \circ \Phi_{L, L, L}^{-1} + \Phi_{L, L, L} \circ c_{L, L \otimes L}) = 0_{L \otimes (L \otimes L), L}$

Remark 3.7. With regard to the notation in Definition 3.6, \otimes is the monoidal product; I is the unit object; Φ is the associator; l and r are the left and right unit maps respectively; and c is the braiding.

Example 3.8. Let \mathbf{Vect}_k be the symmetric monoidal additive category of finite dimensional vector spaces over k . It follows readily from Definition 3.6 that a Lie object (L, b) in \mathbf{Vect}_k is precisely a finite dimensional Lie algebra L over k with Lie bracket $[x, y] := b(x, y)$.

Definition 3.9. A *quasi-Frobenius Lie object* in an additive symmetric monoidal category $(\mathcal{C}, \otimes, I, \Phi, l, r, c)$ is a triple $(L, b, \bar{\beta})$ such that

- (1) (L, b) is a Lie object.
- (2) L has a left dual object L^* (where $\varepsilon : L^* \otimes L \rightarrow I$ and $\eta : I \rightarrow L \otimes L^*$ denote the evaluation and coevaluation morphisms respectively).
- (3) $\bar{\beta} : L \xrightarrow{\sim} L^*$ is an isomorphism such that the induced morphism

$$\beta := \varepsilon \circ (\bar{\beta} \otimes id_L) : L \otimes L \rightarrow I,$$

satisfies

$$\beta + \beta \circ c_{L, L} = 0_{L \otimes L, I}$$

and

$$\beta \circ (b \otimes id_L) \circ [id_{(L \otimes L) \otimes L} + \Phi_{L, L, L}^{-1} \circ c_{L \otimes L, L} + c_{L, L \otimes L} \circ \Phi_{L, L, L}] = 0_{(L \otimes L) \otimes L, I}.$$

If there exists a morphism $\alpha : L \rightarrow I$ such that $\beta = \alpha \circ b$, then $(L, b, \bar{\beta})$ is called a *Frobenius Lie object*.

Example 3.10. Let $(L, b, \bar{\beta})$ be a quasi-Frobenius Lie object in \mathbf{Vect}_k . Then its easy to see that L is a quasi-Frobenius Lie algebra over k with Lie bracket $[x, y] := b(x, y)$ and $\beta : L \otimes L \rightarrow k$ (as defined in (3) of Definition 3.9)

is the nondegenerate 2-cocycle in the Lie algebra cohomology of L . Likewise, a Frobenius Lie object in \mathbf{Vect}_k is just a Frobenius Lie algebra.

Proposition 3.11. *The category $\mathbf{Rep}(\mathfrak{g})$ of finite dimensional left \mathfrak{g} -modules over k is an additive symmetric monoidal category where every object has a left dual and*

- (i) *the monoidal product is the usual tensor product of left \mathfrak{g} -modules and \mathfrak{g} -linear maps*
- (ii) *the unit object is k with the trivial \mathfrak{g} -action*
- (iii) *the associator Φ is the trivial one*
- (iv) *for any object (V, ρ) in $\mathbf{Rep}(\mathfrak{g})$, the left and right morphisms $l_V : k \otimes V \xrightarrow{\sim} V$ and $r_V : V \otimes k \xrightarrow{\sim} V$ are the trivial ones*
- (v) *for objects (V, ρ) , (W, ϕ) in $\mathbf{Rep}(\mathfrak{g})$, the braiding $c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ is simply the linear map that sends $v \otimes w \in V \otimes W$ to $w \otimes v \in W \otimes V$*
- (vi) *the left dual of an object (V, ρ) in $\mathbf{Rep}(\mathfrak{g})$ is the dual representation (V^*, ρ^*) (i.e., $\rho_x^* := -\rho_x^t : V^* \rightarrow V^*$ for $x \in \mathfrak{g}$, where ρ_x^t is the dual or transpose of $\rho_x : V \rightarrow V$)*
- (vii) *the evaluation morphism is $\varepsilon : V^* \otimes V \rightarrow k$, $\varepsilon(\xi, v) := \xi(v)$ and the coevaluation morphism is $\eta : k \rightarrow V \otimes V^*$, $1 \mapsto \sum_i e_i \otimes \delta^i$ where e_i is any basis of V and δ^i is the corresponding dual basis.*

Proof. It is an easy exercise to verify that $(\mathbf{Rep}(\mathfrak{g}), \otimes, k, \Phi, l, r, c)$ satisfies all the axioms of an additive symmetric monoidal category. \square

The next result establishes the categorical formulation of \mathfrak{g} -quasi-Frobenius Lie algebras.

Proposition 3.12. *A quasi-Frobenius Lie object in $\mathbf{Rep}(\mathfrak{g})$ is a \mathfrak{g} -quasi-Frobenius Lie algebra.*

Proof. By definition, a quasi-Frobenius Lie object in $\mathbf{Rep}(\mathfrak{g})$ consists of a representation (\mathfrak{q}, ρ) of \mathfrak{g} together with \mathfrak{g} -linear maps

$$b : \mathfrak{q} \otimes \mathfrak{q} \rightarrow \mathfrak{q}, \quad \bar{\beta} : \mathfrak{q} \xrightarrow{\sim} \mathfrak{q}^*,$$

which satisfy conditions (1) and (3) of Definition 3.9.

We begin by verifying that (\mathfrak{q}, β) is a quasi-Frobenius Lie algebra. To start, note that condition (1) of Definition 3.9 implies that \mathfrak{q} is a Lie algebra with Lie bracket $[u, v] := b(u, v)$. From Definition 3.9, the morphism $\beta : \mathfrak{q} \otimes \mathfrak{q} \rightarrow k$ is given explicitly as

$$\beta(u, v) = \varepsilon(\bar{\beta}(u), v) = \bar{\beta}(u)(v).$$

Condition (3) of Definition 3.9 implies that β is a 2-cocycle of \mathfrak{q} with values in k (where \mathfrak{q} acts trivially on k). Furthermore, since $\bar{\beta} : \mathfrak{q} \xrightarrow{\sim} \mathfrak{q}^*$ is an isomorphism, it follows that β is nondegenerate. Hence, (\mathfrak{q}, β) is a quasi-Frobenius Lie algebra.

Since $\overline{\beta}$ is \mathfrak{g} -linear (being a morphism of $\mathbf{Rep}(\mathfrak{g})$), we have

$$\overline{\beta}(\rho_x(u))(v) = \rho_x^*(\overline{\beta}(u))(v) = -\overline{\beta}(u)(\rho_x(v)), \quad \forall u, v \in \mathfrak{q} \quad (3.11)$$

where we recall that $\rho_x^* := -\rho_x^t$. Expressing the left and right most sides of (3.11) in terms of β gives

$$\beta(\rho_x(u), v) = -\beta(u, \rho_x(v)),$$

which proves the \mathfrak{g} -invariance of β , that is, $\beta(\rho_x(u), v) + \beta(u, \rho_x(v)) = 0$.

Since b is also \mathfrak{g} -linear, we also have

$$\begin{aligned} \rho_x([u, v]) &= \rho_x(b(u \otimes v)) \\ &= b(\overline{\rho}_x(u \otimes v)) \\ &= b(\rho_x(u) \otimes v) + b(u \otimes \rho_x(v)) \\ &= [\rho_x(u), v] + [u, \rho_x(v)], \end{aligned}$$

where $\overline{\rho}_x$ in the second equality denotes the induced left \mathfrak{g} -module structure on $\mathfrak{q} \otimes \mathfrak{q}$. Hence, $(\mathfrak{q}, \beta, \rho)$ is a \mathfrak{g} -quasi-Frobenius Lie algebra. \square

4. THE GEOMETRY OF \mathfrak{g} -QUASI-FROBENIUS LIE ALGEBRAS

4.1. G -Symplectic Lie groups.

Definition 4.1. Let G be a Lie group. A G -symplectic Lie group is a triple (Q, ω, φ) where (Q, ω) is a symplectic Lie group and

$$\varphi : G \times Q \rightarrow Q, \quad (g, q) \mapsto \varphi_g(q) := \varphi(g, q)$$

is a smooth left action on Q such that $\varphi_g : (Q, \omega) \rightarrow (Q, \omega)$ is an isomorphism of symplectic Lie groups.

Notation 4.2. When dealing with multiple Lie groups, we will denote the identity element of each group simply as e as opposed to e_G for G , e_Q for Q , and so on when there is no risk of confusion.

Proposition 4.3. Let (Q, ω, φ) be a G -symplectic Lie group with action

$$\varphi : G \times Q \rightarrow Q, \quad (g, q) \mapsto \varphi_g(q) := \varphi(g, q).$$

Define

$$\varphi' : G \rightarrow GL(\mathfrak{q}), \quad g \mapsto \varphi'_g := (\varphi_g)_{*,e} : \mathfrak{q} \rightarrow \mathfrak{q}$$

$$\varphi'' : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{q}), \quad x \mapsto \varphi''_x := (\varphi')_{*,e}(x) : \mathfrak{q} \rightarrow \mathfrak{q}.$$

Then

- (i) φ' is a representation of G on \mathfrak{q} such that $\varphi'_g \in \text{Aut}(\mathfrak{q}, \omega_e)$ for all $g \in G$.
- (ii) $(\mathfrak{q}, \omega_e, \varphi'')$ is a \mathfrak{g} -quasi-Frobenius Lie algebra.

Proof. Since φ is a left action of G on Q and $\varphi_g(e) = e$ for all $g \in G$, we have

$$\begin{aligned}\varphi'_g \circ \varphi'_h &= (\varphi_g)_{*,e} \circ (\varphi_h)_{*,e} \\ &= (\varphi_g \circ \varphi_h)_{*,e} \\ &= (\varphi_{gh})_{*,e} \\ &= \varphi'_{gh}.\end{aligned}$$

Hence, φ' is a representation of G on \mathfrak{q} . Furthermore, since $\varphi_g : Q \rightarrow Q$ is both a Lie group isomorphism and a symplectomorphism, it follows that $\varphi'_g : \mathfrak{q} \rightarrow \mathfrak{q}$ is a Lie algebra isomorphism and

$$\omega_e(u, v) = ((\varphi_g)^*\omega)_e(u, v) = \omega_e((\varphi_g)_{*,e}(u), (\varphi_g)_{*,e}(v)) = \omega_e(\varphi'_g(u), \varphi'_g(v)),$$

which shows that $\varphi'_g \in \text{Aut}(\mathfrak{q}, \omega_e)$ for all $g \in G$. This proves (i).

Statement (ii) follows from an application of Proposition 3.3 to the quasi-Frobenius Lie algebra (\mathfrak{q}, ω_e) with Lie group homomorphism $\varphi' : G \rightarrow \text{Aut}(\mathfrak{q}, \omega_e) \subset GL(\mathfrak{q})$. This completes the proof. \square

Remark 4.4. We will refer to $(\mathfrak{q}, \omega_e, \varphi'')$ in Proposition 4.3 as the \mathfrak{g} -quasi-Frobenius Lie algebra associated to the G -symplectic Lie group (Q, ω, φ) .

The next result provides a means of constructing G -symplectic Lie groups.

Proposition 4.5. *Let (Q, ω) be a simply connected symplectic Lie group, let G be a Lie group, and let $\rho : G \rightarrow \text{Aut}(\mathfrak{q}, \omega_e)$, $g \mapsto \rho_g$ be a Lie group homomorphism. Then there exists a unique smooth left- G action*

$$\hat{\rho} : G \times Q \rightarrow Q, \quad (g, q) \mapsto \hat{\rho}_g(q),$$

such that $(Q, \omega, \hat{\rho})$ is a G -symplectic Lie group and $(\hat{\rho}_g)_{,e} = \rho_g$. In particular, if G is any Lie subgroup of $\text{Aut}(\mathfrak{q}, \omega_e)$ and $G \neq \{e\}$, then (Q, ω) admits the structure of a G -symplectic Lie group with a nontrivial G -action.*

Proof. Let $\rho : G \rightarrow \text{Aut}(\mathfrak{q}, \omega_e)$, $g \mapsto \rho_g$ be a Lie group homomorphism. Since Q is simply connected and $\rho_g \in \text{Aut}(\mathfrak{q}, \omega_e)$ for all $g \in G$, it follows from Proposition 2.17 that there exists a unique homomorphism of symplectic Lie groups

$$\hat{\rho}_g : (Q, \omega) \rightarrow (Q, \omega)$$

such that $(\hat{\rho}_g)_{*,e} = \rho_g$ for all $g \in G$. Furthermore, for $g, h \in G$, we have

$$\begin{aligned}(\hat{\rho}_g \circ \hat{\rho}_h)_{*,e} &= (\hat{\rho}_g)_{*,e} \circ (\hat{\rho}_h)_{*,e} \\ &= \rho_g \circ \rho_h \\ &= \rho_{gh} \\ &= (\hat{\rho}_{gh})_{*,e}.\end{aligned}\tag{4.1}$$

Since $\hat{\rho}_g \circ \hat{\rho}_h$ and $\hat{\rho}_{gh}$ are Lie group homomorphisms and Q is connected, equation (4.1) implies that

$$\hat{\rho}_g \circ \hat{\rho}_h = \hat{\rho}_{gh}.\tag{4.2}$$

Hence,

$$\widehat{\rho} : G \times Q \rightarrow Q, \quad (g, q) \mapsto \widehat{\rho}_g(q)$$

is a left (not necessarily smooth) G -action. We now show that $\widehat{\rho}$ is smooth. To do this, set $\widehat{\rho}(g, q) = \widehat{\rho}_g(q)$ for $g \in G$, $q \in Q$ and let U be an open neighborhood of $0 \in \mathfrak{q}$ such that

$$\exp|_U : U \xrightarrow{\sim} \exp(U)$$

is a diffeomorphism. The naturality of the exponential map implies that

$$\widehat{\rho}(g, q) = \exp \circ \rho_g \circ (\exp|_U)^{-1}(q), \quad \forall (g, q) \in G \times \exp(U). \quad (4.3)$$

Since the right side of (4.3) is smooth on $G \times \exp(U)$, it follows that $\widehat{\rho}|_{G \times \exp(U)}$ is also smooth. Now fix an arbitrary element q_0 of Q and define

$$f : G \rightarrow Q, \quad g \mapsto \widehat{\rho}(g, q_0).$$

We now show that f is smooth. Since Q is connected, $\exp(U)$ generates Q . Hence, there exists $q_{0,1}, \dots, q_{0,k} \in \exp(U)$ such that

$$q_0 = q_{0,1}q_{0,2} \cdots q_{0,k}.$$

Since $\widehat{\rho}_g : Q \rightarrow Q$ is a Lie group homomorphism for all $g \in G$, we have

$$f(g) := \widehat{\rho}(g, q_0) = \widehat{\rho}(g, q_{0,1})\widehat{\rho}(g, q_{0,2}) \cdots \widehat{\rho}(g, q_{0,k}) \in Q. \quad (4.4)$$

Since $(g, q_{0,i}) \in G \times \exp(U)$ for $i = 1, \dots, k$, it follows that the right side of (4.4) depends smoothly on g . Hence, f is smooth. Now, for all $(g, q) \in G \times (q_0 \exp(U))$, we have

$$\begin{aligned} \widehat{\rho}(g, q) &= \widehat{\rho}(g, q_0 q_0^{-1} q) \\ &= \widehat{\rho}(g, q_0) \widehat{\rho}(g, q_0^{-1} q) \\ &= f(g) [(\widehat{\rho}|_{G \times \exp(U)}) \circ (id_G \times l_{q_0^{-1}})(g, q)], \end{aligned} \quad (4.5)$$

where $l_{q_0^{-1}} : Q \rightarrow Q$ is left translation by q_0^{-1} . Since f and $\widehat{\rho}|_{G \times \exp(U)}$ are both smooth, it follows that the right side of (4.5) is smooth on $G \times (q_0 \exp(U))$. Hence, $\widehat{\rho}|_{G \times (q_0 \exp(U))}$ is smooth. Since $q_0 \in Q$ is arbitrary, it follows that $\widehat{\rho}$ is smooth on $G \times Q$. This completes the proof. \square

We now illustrate Proposition 4.5 with a simple example:

Example 4.6. Let Q be the 2-dimensional non-abelian Lie group

$$Q = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}. \quad (4.6)$$

Note that Q is simply connected, being diffeomorphic to $\mathbb{R}_+ \times \mathbb{R}$. The associated Lie algebra is

$$\mathfrak{q} = \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & 0 \end{pmatrix} \mid \bar{a}, \bar{b} \in \mathbb{R} \right\}. \quad (4.7)$$

A convenient basis for \mathfrak{q} is then

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (4.8)$$

where we note that

$$[e_1, e_2] = e_2. \quad (4.9)$$

Let $\alpha : \mathfrak{q} \rightarrow \mathbb{R}$ be the linear map defined by $\alpha(e_1) = 0$ and $\alpha(e_2) = 1$. Then (\mathfrak{q}, α) is a Frobenius Lie algebra. Let $\tilde{\beta}$ be the left-invariant symplectic form on Q defined by $\tilde{\beta}_e = \beta$, where $\beta(u, v) := \alpha([u, v])$ for $u, v \in \mathfrak{q}$.

For $\lambda \in \mathbb{R}$, let $\rho_\lambda : \mathfrak{q} \rightarrow \mathfrak{q}$ be the linear isomorphism defined by

$$\rho_\lambda(e_1) := e_1 + \lambda e_2, \quad \rho_\lambda(e_2) := e_2.$$

Then it is a straightforward exercise to show that $\rho_\lambda \in \text{Aut}(\mathfrak{q}, \omega_e)$ and

$$\rho : \mathbb{R} \xrightarrow{\sim} \text{Aut}(\mathfrak{q}, \omega_e), \quad \lambda \mapsto \rho_\lambda$$

is a Lie group isomorphism. Proposition 4.5 implies that (Q, ω) admits the structure of an \mathbb{R} -symplectic Lie group with unique action $\hat{\rho} : \mathbb{R} \times Q \rightarrow Q$ satisfying $(\hat{\rho}_\lambda)_{*,e} = \rho_\lambda$.

We now compute the action $\hat{\rho}$ explicitly. Let $u \in \mathfrak{q}$. Then

$$u = \bar{a}e_1 + \bar{b}e_2 = \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & 0 \end{pmatrix} \quad (4.10)$$

for some $a, b \in \mathbb{R}$. Using the naturality of the exponential map, we have

$$\hat{\rho}_\lambda \circ \exp(u) = \exp \circ \rho_\lambda(u). \quad (4.11)$$

A direct calculation shows that

$$\exp(u) = \begin{pmatrix} e^{\bar{a}} & \mu(\bar{a})\bar{b} \\ 0 & 1 \end{pmatrix}, \quad (4.12)$$

where $\mu : \mathbb{R} \rightarrow \mathbb{R}_+$ is the nonzero smooth function given by $\mu(t) = \frac{1}{t}(e^t - 1)$ for $t \neq 0$ and $\mu(0) = 1$. Note that every element of Q is in the image of the exponential map. Indeed, given

$$q = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

for $a > 0$, $b \in \mathbb{R}$, one simply sets $\bar{a} = \ln a$ and $\bar{b} = b/\mu(\ln a)$ in (4.12) to obtain $\exp(u) = q$. The left side of (4.11) is

$$\exp \circ \rho_\lambda(u) = \exp \begin{pmatrix} \bar{a} & \lambda\bar{a} + \bar{b} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{\bar{a}} & \mu(\bar{a})(\lambda\bar{a} + \bar{b}) \\ 0 & 1 \end{pmatrix}. \quad (4.13)$$

Hence,

$$\hat{\rho}_\lambda \begin{pmatrix} e^{\bar{a}} & \mu(\bar{a})\bar{b} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\bar{a}} & \mu(\bar{a})(\lambda\bar{a} + \bar{b}) \\ 0 & 1 \end{pmatrix}. \quad (4.14)$$

Setting $\bar{a} = \ln a$ and $\bar{b} = b/\mu(\ln a)$ for $a > 0$ and $b \in \mathbb{R}$, we obtain

$$\hat{\rho}_\lambda \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \lambda(a - 1) + b \\ 0 & 1 \end{pmatrix}. \quad (4.15)$$

Since (Q, ω, φ) is an \mathbb{R} -symplectic Lie group by Proposition 4.5, $\widehat{\rho}_\lambda$ is both a Lie group isomorphism and a symplectomorphism of (Q, ω) which satisfies $(\widehat{\rho}_\lambda)_{*,e} = \rho_\lambda$.

In anticipation of the next section, we introduce the following definition:

Definition 4.7. Let (Q, ω, φ) and (R, τ, χ) be G -symplectic Lie groups. A homomorphism of G -symplectic Lie groups from (Q, ω, φ) to (R, τ, χ) is a homomorphism

$$\Psi : (Q, \omega) \rightarrow (R, \tau)$$

of symplectic Lie groups which is also G -equivariant, that is, $\Psi(\varphi_g(q)) = \chi_g(\Psi(q))$ for all $g \in G$ and $q \in Q$.

4.2. The Equivalence. In this section, we show that the category of finite dimensional \mathfrak{g} -quasi-Frobenius Lie algebras is equivalent to the category of simply connected G -symplectic Lie groups, where G is also simply connected. We begin with the following result.

Proposition 4.8. Let $\Psi : (Q, \omega, \varphi) \rightarrow (R, \tau, \chi)$ be a homomorphism of G -symplectic Lie groups. Then

$$\Psi_{*,e} : (\mathfrak{q}, \omega_e, \varphi'') \rightarrow (\mathfrak{r}, \tau_e, \chi'')$$

is a homomorphism of \mathfrak{g} -quasi-Frobenius Lie algebras, where φ'' and χ'' are defined as in Proposition 4.3.

Proof. By definition, $\Psi : (Q, \omega) \rightarrow (R, \tau)$ is a homomorphism of symplectic Lie groups. This implies that

$$\Psi_{*,e} : (\mathfrak{q}, \omega_e) \rightarrow (\mathfrak{r}, \tau_e)$$

is a homomorphism of quasi-Frobenius Lie algebras. It only remains to show that $\Psi_{*,e}$ is \mathfrak{g} -equivariant. Since Ψ is G -equivariant, we have

$$\Psi \circ \varphi_g = \chi_g \circ \Psi, \quad \forall g \in G.$$

This in turn implies that

$$\Psi_{*,e} \circ \varphi'_g = \chi'_g \circ \Psi_{*,e}, \quad \forall g \in G, \tag{4.16}$$

where $\varphi'_g := (\varphi_g)_{*,e} : \mathfrak{q} \rightarrow \mathfrak{q}$ and $\chi'_g := (\chi_g)_{*,e} : \mathfrak{r} \rightarrow \mathfrak{r}$. Let $x \in \mathfrak{g}$ and set $g = \exp(tx)$ in (4.16). Applying $\frac{d}{dt}|_{t=0}$ to both sides then gives

$$\Psi_{*,e} \circ \varphi''_x = \chi''_x \circ \Psi_{*,e}. \tag{4.17}$$

This in turn completes the proof. \square

Lemma 4.9. Let $(\mathfrak{q}, \beta, \phi)$ be a \mathfrak{g} -quasi-Frobenius Lie algebra and let G be the simply connected Lie group whose Lie algebra is \mathfrak{g} . Then there exists a unique Lie group homomorphism $f : G \rightarrow GL(\mathfrak{q})$, $g \mapsto f_g$ such that $f_{*,e} = \phi$ and $f_g \in \text{Aut}(\mathfrak{q}, \beta)$ for all $g \in G$.

Proof. Since G is simply connected and $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{q})$ is a Lie algebra map, there exists a unique Lie group homomorphism $f : G \rightarrow GL(\mathfrak{q})$ such that $f_{*,e} = \phi$. We now show that $f_g \in \text{Aut}(\mathfrak{q}, \beta)$ for all $g \in G$. Fix $x \in \mathfrak{g}$. To simplify notation, let

$$f_t := f_{\exp(tx)} : \mathfrak{q} \rightarrow \mathfrak{q}. \quad (4.18)$$

Define $A : \mathbb{R} \times \mathfrak{q} \times \mathfrak{q} \rightarrow \mathbb{R}$ by

$$A(t, u, v) := \beta(f_t(u), f_t(v)) - \beta(u, v). \quad (4.19)$$

Since $f_{*,e} = \phi$ and $(\mathfrak{q}, \beta, \phi)$ is a \mathfrak{g} -quasi-Frobenius Lie algebra, we have

$$\frac{d}{dt}\bigg|_{t=0} A(t, u, v) = \beta(\phi_x(u), v) + \beta(u, \phi_x(v)) = 0, \quad \forall u, v \in \mathfrak{q}. \quad (4.20)$$

Furthermore, since f is a group homomorphism and

$$\exp((t+s)x) = \exp(tx)\exp(sx),$$

we have

$$A(t+s, u, v) = A(t, f_s(u), f_s(v)) + A(s, u, v), \quad \forall u, v \in \mathfrak{q}. \quad (4.21)$$

Equations (4.20) and (4.21) imply

$$\frac{d}{dt}\bigg|_{t=s} A(t, u, v) = \frac{d}{dt}\bigg|_{t=0} A(t+s, u, v) = 0 + 0 = 0. \quad (4.22)$$

Hence, for fixed $u, v \in \mathfrak{q}$, $A(t, u, v)$ is a constant. Since $A(0, u, v) = 0$, it follows that $A(t, u, v) = 0$ for all $t \in \mathbb{R}$. Hence,

$$\beta(f_t(u), f_t(v)) = \beta(u, v), \quad \forall t \in \mathbb{R}. \quad (4.23)$$

In particular,

$$\beta(f_{\exp(x)}(u), f_{\exp(x)}(v)) = \beta(u, v). \quad (4.24)$$

Now define $B : \mathbb{R} \times \mathfrak{q} \times \mathfrak{q} \times \mathfrak{q} \rightarrow \mathfrak{q}$ by

$$B(t, u, v, w) = \beta([f_t(u), f_t(v)] - f_t([u, v]), f_t(w)). \quad (4.25)$$

Equation (4.23) implies that

$$B(t, u, v, w) = \beta([f_t(u), f_t(v)], f_t(w)) - \beta([u, v], w). \quad (4.26)$$

Using (4.26) and the fact that $(\mathfrak{q}, \beta, \phi)$ is a \mathfrak{g} -quasi-Frobenius Lie algebra, we have

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} B(t, u, v, w) &= \beta([\phi_x(u), v], w) + \beta([u, \phi_x(v)], w) + \beta([u, v], \phi_x(w)) \\ &= \beta(\phi_x([u, v]), w) + \beta([u, v], \phi_x(w)) \\ &= 0, \quad \forall u, v, w. \end{aligned} \quad (4.27)$$

From (4.26), we also have

$$\begin{aligned} B(t+s, u, v, w) &= \beta([f_t(f_s(u)), f_t(f_s(v))], f_t(f_s(w))) - \beta([u, v], w) \\ &= B(t, f_s(u), f_s(v), f_s(w)) + \beta([f_s(u), f_s(v)], f_s(w)) - \beta([u, v], w) \end{aligned} \quad (4.28)$$

Equations (4.27) and (4.28) now imply

$$\frac{d}{dt}\Big|_{t=s} B(t, u, v, w) = \frac{d}{dt}\Big|_{t=0} B(t + s, u, v, w) = 0 + 0 + 0 = 0. \quad (4.29)$$

From (4.29), it follows that for fixed u, v, w , $B(t, u, v, w)$ is a constant for all $t \in \mathbb{R}$. Hence, $B(t, u, v, w) = B(0, u, v, w) = 0$ for all $t \in \mathbb{R}$ and $u, v, w \in \mathfrak{q}$. In particular,

$$B(1, u, v, w) = \beta([f_1(u), f_1(v)] - f_1([u, v]), f_1(w)) = 0, \quad \forall u, v, w \in \mathfrak{q}. \quad (4.30)$$

Since β is non-degenerate and $f_1 := f_{\exp(x)} \in GL(\mathfrak{q})$, it follows that

$$f_{\exp(x)}([u, v]) = [f_{\exp(x)}(u), f_{\exp(x)}(v)]. \quad (4.31)$$

Since G is connected, $x \in \mathfrak{g}$ is arbitrary, and f is a group homomorphism, equations (4.24) and (4.31) imply that

$$\beta(f_g(u), f_g(v)) = \beta(u, v), \quad f_g([u, v]) = [f_g(u), f_g(v)] \quad (4.32)$$

for all $g \in G$. Hence, $f_g \in \text{Aut}(\mathfrak{q}, \beta)$ for all $g \in G$. This completes the proof. \square

Proposition 4.10. *Let $(\mathfrak{q}, \beta, \phi)$ be a \mathfrak{g} -quasi-Frobenius Lie algebra. Let G and Q be the simply connected Lie groups associated to \mathfrak{g} and \mathfrak{q} respectively and let $\tilde{\beta} \in \Omega^2(Q)$ be the left-invariant 2-form associated to β . Then there exists a unique left action $\bar{\phi} : G \times Q \rightarrow Q$ such that $(Q, \tilde{\beta}, \bar{\phi})$ is a G -symplectic Lie group whose associated \mathfrak{g} -quasi-Frobenius Lie algebra is*

$$(\mathfrak{q}, \tilde{\beta}_e, \bar{\phi}'') = (\mathfrak{q}, \beta, \phi),$$

where $\bar{\phi}''$ is defined as in Proposition 4.3.

Proof. By Proposition 2.13, $(Q, \tilde{\beta})$ is a symplectic Lie group. Since G is simply connected, Lemma 4.9 shows that there exists a unique Lie group homomorphism

$$f : G \rightarrow GL(\mathfrak{q}), \quad g \mapsto f_g$$

such that $f_{*,e} = \phi : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{q})$ and $f_g \in \text{Aut}(\mathfrak{q}, \beta)$ for all $g \in G$. Since Q is simply connected, Proposition 4.5 shows that there exists a unique smooth left G -action

$$\bar{\phi} : G \times Q \rightarrow Q, \quad (g, q) \mapsto \bar{\phi}_g(q)$$

such that $(Q, \tilde{\beta}, \bar{\phi})$ is a G -symplectic Lie group and $(\bar{\phi}_g)_{*,e} = f_g$. Setting $\bar{\phi}'_g := (\bar{\phi}_g)_{*,e}$ as in Proposition 4.3, we have

$$\bar{\phi}'' := \bar{\phi}'_{*,e} = f_{*,e} = \phi.$$

This completes the proof. \square

Proposition 4.11. *Let $\psi : (\mathfrak{q}, \beta, \phi) \rightarrow (\mathfrak{r}, \sigma, \mu)$ be a homomorphism of \mathfrak{g} -quasi-Frobenius Lie algebras. Let G be the simply connected Lie group whose Lie algebra is \mathfrak{g} and let $(Q, \tilde{\beta}, \bar{\phi})$ and $(R, \tilde{\sigma}, \bar{\mu})$ be the simply connected G -symplectic Lie groups associated to $(\mathfrak{q}, \beta, \phi)$ and $(\mathfrak{r}, \sigma, \mu)$ respectively by*

Proposition 4.10. *Then there exists a unique homomorphism of G -symplectic Lie groups*

$$\widehat{\psi} : (Q, \widetilde{\beta}, \overline{\phi}) \rightarrow (R, \widetilde{\sigma}, \overline{\mu})$$

such that $\widehat{\psi}_{*,e} = \psi$.

Proof. By Proposition 2.17, there exists a unique homomorphism of symplectic Lie groups $\widehat{\psi} : (Q, \widetilde{\beta}) \rightarrow (R, \widetilde{\sigma})$ such that $\widehat{\psi}_{*,e} = \psi$. We now verify that $\widehat{\psi}$ is G -equivariant.

Let $\overline{\phi}' : G \rightarrow \text{Aut}(\mathfrak{q}, \beta)$, $g \mapsto \overline{\phi}'_g$ and $\overline{\mu}' : G \rightarrow \text{Aut}(\mathfrak{r}, \sigma)$, $g \mapsto \overline{\mu}'_g$ be defined as in Proposition 4.3. Fix $x \in \mathfrak{g}$. To simplify notation, let

$$\overline{\phi}'_t := \overline{\phi}'_{\exp(tx)}, \quad \overline{\mu}'_t := \overline{\mu}'_{\exp(tx)}.$$

Define $B : \mathbb{R} \times \mathfrak{q} \times \mathfrak{r} \rightarrow \mathbb{R}$ by

$$\begin{aligned} B(t, u, v) &:= \sigma(\psi \circ \overline{\phi}'_t(u) - \overline{\mu}'_t \circ \psi(u), \overline{\mu}'_t(v)) \\ &= \sigma(\psi \circ \overline{\phi}'_t(u), \overline{\mu}'_t(v)) - \sigma(\overline{\mu}'_t \circ \psi(u), \overline{\mu}'_t(v)) \\ &= \sigma(\psi \circ \overline{\phi}'_t(u), \overline{\mu}'_t(v)) - \sigma(\psi(u), v), \end{aligned} \quad (4.33)$$

where the third equality follows from the fact that $\overline{\mu}'_t \in \text{Aut}(\mathfrak{r}, \sigma)$. Hence,

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} B(t, u, v) &= \sigma(\psi \circ \phi_x(u), v) + \sigma(\psi(u), \mu_x(v)) \\ &= \sigma(\mu_x \circ \psi(u), v) + \sigma(\psi(u), \mu_x(v)) \\ &= 0, \quad \forall u \in \mathfrak{q}, v \in \mathfrak{r} \end{aligned} \quad (4.34)$$

where the second equality follows from the fact that ψ is \mathfrak{g} -equivariant (i.e., $\psi \circ \phi_x = \mu_x \circ \psi$) and the third equality follows from the fact that $(\mathfrak{r}, \sigma, \mu)$ is a \mathfrak{g} -quasi-Frobenius Lie algebra with 2-cocycle σ and \mathfrak{g} -action μ . Next note that

$$B(t+s, u, v) = B(t, \overline{\phi}'_s(u), \overline{\mu}'_s(v)) + \sigma(\psi(\overline{\phi}'_s(u)), \overline{\mu}'_s(v)) - \sigma(\psi(u), v) \quad (4.35)$$

Hence,

$$\frac{d}{dt} \Big|_{t=s} B(t, u, v) = \frac{d}{dt} \Big|_{t=0} B(t+s, u, v) = 0 + 0 - 0 = 0, \quad (4.36)$$

where the first zero follows from (4.34). Hence,

$$B(t, u, v) = B(0, u, v) = 0, \quad \forall t \in \mathbb{R}, u \in \mathfrak{q}, v \in \mathfrak{r}. \quad (4.37)$$

In particular, $B(1, u, v) = 0$ for all $u, v \in \mathfrak{r}$. Since σ is nondegenerate and $\overline{\mu}'_t : \mathfrak{r} \rightarrow \mathfrak{r}$ is also a linear isomorphism for all t , it follows that

$$\psi \circ \overline{\phi}'_t = \overline{\mu}'_t \circ \psi, \quad \forall t \in \mathbb{R}. \quad (4.38)$$

In particular, we have

$$\psi \circ \overline{\phi}'_{\exp(x)} = \overline{\mu}'_{\exp(x)} \circ \psi. \quad (4.39)$$

Since $x \in \mathfrak{g}$ was arbitrary, (4.39) must hold for all $x \in \mathfrak{g}$. Since G is connected, every element $g \in G$ is of the form $g = \exp(x_1) \cdots \exp(x_k)$ for

some $x_i \in \mathfrak{g}$, $i = 1, \dots, k$. It follows from this and the fact that $\overline{\phi}'$ and $\overline{\mu}'$ are group homomorphisms that

$$\psi \circ \overline{\phi}'_g = \overline{\mu}'_g \circ \psi, \quad \forall g \in G. \quad (4.40)$$

Equation (4.40) combined with the fact that (1) Q is connected, (2) $\widehat{\psi} \circ \overline{\phi}_g$ and $\overline{\mu}_g \circ \widehat{\psi}$ are both Lie group homomorphisms $\forall g \in G$, and (3)

$$(\widehat{\psi} \circ \overline{\phi}_g)_{*,e} = \psi \circ \overline{\phi}'_g = \overline{\mu}'_g \circ \psi = (\overline{\mu}_g \circ \widehat{\psi})_{*,e}, \quad \forall g \in G \quad (4.41)$$

imply that $\widehat{\psi} \circ \overline{\phi}_g = \overline{\mu}_g \circ \widehat{\psi}$ for all $g \in G$. In other words, $\widehat{\psi}$ is G -equivariant and this completes the proof. \square

We conclude the paper with the following generalization of Theorem 2.18.

Theorem 4.12. *Let G be a simply connected Lie group and let $G\text{-SCSLG}$ be the category of simply connected G -symplectic Lie groups and let $\mathfrak{g}\text{-qFLA}$ be the category of finite dimensional \mathfrak{g} -quasi-Frobenius Lie algebras. Let \widehat{F} be the functor from $G\text{-SCSLG}$ to $\mathfrak{g}\text{-qFLA}$ which sends the object (Q, ω, φ) to $(\mathfrak{q}, \omega_e, \varphi'')$, where φ'' is defined as in Proposition 4.3 and the morphism $\Psi : (Q, \omega, \varphi) \rightarrow (R, \tau, \chi)$ to*

$$\Psi_{*,e} : (\mathfrak{q}, \omega_e, \varphi'') \mapsto (\mathfrak{r}, \tau_e, \chi'').$$

Then \widehat{F} is an equivalence of categories.

Proof. Theorem 4.12 follows from Theorem 2.18, Proposition 4.3, Proposition 4.8, Proposition 4.10, and Proposition 4.11. \square

5. $D(\mathfrak{g})$ -QUASI-FROBENIUS LIE ALGEBRAS

Let (\mathfrak{g}, γ) be a finite dimensional Lie bialgebra. We begin with the following observation:

Proposition 5.1. *Let V be a vector space over k and let $\rho : D(\mathfrak{g}) \rightarrow \mathfrak{gl}(V)$ be a linear map (not necessarily a representation). Define*

$$\varphi := \rho|_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{gl}(V), \quad \psi := \rho|_{\mathfrak{g}^*} : \mathfrak{g}^* \rightarrow \mathfrak{gl}(V).$$

The following statements are equivalent.

- (i) ρ is a representation of $D(\mathfrak{g})$ on V .
- (ii) φ and ψ are representations of \mathfrak{g} and \mathfrak{g}^* on V which satisfy

$$\psi_{ad_x^* \xi} - \varphi_{ad_\xi^* x} = \varphi_x \circ \psi_\xi - \psi_\xi \circ \varphi_x, \quad \forall x \in \mathfrak{g}, \xi \in \mathfrak{g}^*. \quad (5.1)$$

Proof. (i) \Rightarrow (ii). Since ρ is a representation of $D(\mathfrak{g})$ on V , it follows immediately that φ and ψ must be representations of \mathfrak{g} and \mathfrak{g}^* on V respectively. For (5.1), we note that

$$[x, \xi] = ad_x^* \xi - ad_\xi^* x \quad \forall x \in \mathfrak{g}, \xi \in \mathfrak{g}^*.$$

Since ρ is a representation and $\varphi := \rho|_{\mathfrak{g}}$ and $\psi := \rho|_{\mathfrak{g}^*}$, we have

$$\psi_{ad_x^* \xi} - \varphi_{ad_\xi^* x} = \rho_{[x, \xi]} = \rho_x \rho_\xi - \rho_\xi \rho_x = \varphi_x \psi_\xi - \psi_\xi \varphi_x,$$

which proves (5.1).

(i) \Leftarrow (ii). Let $a = x + \xi \in D(\mathfrak{g})$. Then

$$\begin{aligned}
 \rho_{[x+\xi, y+\eta]} &= \rho_{[x, y]} + \rho_{[x, \eta]} + \rho_{[\xi, y]} + \rho_{[\xi, \eta]} \\
 &= \varphi_{[x, y]} + \psi_{ad_x^* \eta} - \varphi_{ad_\eta^* x} + \varphi_{ad_\xi^* y} - \psi_{ad_y^* \xi} + \psi_{[\xi, \eta]} \\
 &= \varphi_x \circ \varphi_y - \varphi_y \circ \varphi_x + \varphi_x \circ \psi_\eta - \psi_\eta \circ \varphi_x \\
 &\quad + \psi_\xi \circ \varphi_y - \varphi_y \circ \psi_\xi + \psi_\xi \circ \psi_\eta - \psi_\eta \circ \psi_\xi \\
 &= (\varphi_x + \psi_\xi) \circ (\varphi_y + \psi_\eta) - (\varphi_y + \psi_\eta) \circ (\varphi_x + \psi_\xi) \\
 &= \rho_{x+\xi} \circ \rho_{y+\eta} - \rho_{y+\eta} \circ \rho_{x+\xi}.
 \end{aligned}$$

This proves that $\rho : D(\mathfrak{g}) \rightarrow \mathfrak{gl}(V)$ is a representation of $D(\mathfrak{g})$ on V . \square

Proposition 5.2. *Let (\mathfrak{q}, β) be a quasi-Frobenius Lie algebra and let $\rho : D(\mathfrak{g}) \rightarrow \mathfrak{gl}(\mathfrak{q})$ be a linear map (not necessarily a representation). Define $\varphi := \rho|_{\mathfrak{g}}$ and $\psi := \rho|_{\mathfrak{g}^*}$. Then $(\mathfrak{q}, \beta, \rho)$ is a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra iff the following conditions are satisfied:*

- (a) $\psi_{ad_x^* \xi} - \varphi_{ad_\xi^* x} = \varphi_x \circ \psi_\xi - \psi_\xi \circ \varphi_x, \quad \forall x \in \mathfrak{g}, \xi \in \mathfrak{g}^*$
- (b) $(\mathfrak{q}, \beta, \varphi)$ is a \mathfrak{g} -quasi-Frobenius Lie algebra.
- (b) $(\mathfrak{q}, \beta, \psi)$ is a \mathfrak{g}^* -quasi-Frobenius Lie algebra.

Proof. By Proposition 5.1, ρ is left $D(\mathfrak{g})$ -module structure on \mathfrak{q} iff φ and ψ are left \mathfrak{g} and \mathfrak{g}^* -module structures on \mathfrak{q} respectively which satisfy condition (a). Since $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space, it follows that $\rho : D(\mathfrak{g}) \rightarrow \mathfrak{gl}(\mathfrak{q})$ satisfies conditions (i) and (ii) of Definition 3.1 iff $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{q})$ and $\psi : \mathfrak{g}^* \rightarrow \mathfrak{gl}(\mathfrak{q})$ both satisfy conditions (i) and (ii) of Definition 3.1. This completes the proof. \square

Proposition 5.3. *Let \mathfrak{g} be a finite dimensional quasitriangular Lie bialgebra with r -matrix $r = \sum_i a_i \otimes b_i$. Let $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, $x \mapsto \varphi(x)$ be a representation of \mathfrak{g} on V . Define $\psi : \mathfrak{g}^* \rightarrow \mathfrak{gl}(V)$, $\xi \mapsto \psi(\xi)$ by*

$$\psi(\xi) := \sum_i \xi(a_i) \varphi(b_i), \quad \forall \xi \in \mathfrak{g}^*. \quad (5.2)$$

Then ψ is a representation of \mathfrak{g}^ on V .*

Proof. We need to show that

$$\psi([\xi, \eta]) = \psi(\xi)\psi(\eta) - \psi(\eta)\psi(\xi). \quad (5.3)$$

We now expand the left side of (5.3):

$$\begin{aligned}
 \psi([\xi, \eta]) &= \sum_j [\xi, \eta](a_j) \varphi(b_j) \\
 &= \sum_j (\xi \otimes \eta)((\delta r)(a_j)) \varphi(b_j) \\
 &= \sum_{i,j} \xi([a_j, a_i]) \eta(b_i) \varphi(b_j) + \sum_{i,j} \xi(a_i) \eta([a_j, b_i]) \varphi(b_j). \quad (5.4)
 \end{aligned}$$

The right side of (5.3) expands as

$$\begin{aligned}\psi(\xi)\psi(\eta) - \psi(\eta)\psi(\xi) &= \sum_{i,j} \xi(a_i)\eta(a_j)\varphi(b_i)\varphi(b_j) - \sum_{i,j} \eta(a_j)\xi(a_i)\varphi(b_j)\varphi(b_i) \\ &= \sum_{i,j} \xi(a_i)\eta(a_j)\varphi([b_i, b_j]).\end{aligned}\quad (5.5)$$

The CYBE can be rewritten as

$$\sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j] = \sum_{i,j} [a_j, a_i] \otimes b_i \otimes b_j + \sum_{i,j} a_i \otimes [a_j, b_i] \otimes b_j. \quad (5.6)$$

Applying $\xi \otimes \eta \otimes \varphi$ to both sides of (5.6) gives

$$\sum_{i,j} \xi(a_i)\eta(a_j)\varphi([b_i, b_j]) = \sum_{i,j} \xi([a_j, a_i])\eta(b_i)\varphi(b_j) + \sum_{i,j} \xi(a_i)\eta([a_j, b_i])\varphi(b_j). \quad (5.7)$$

Equations (5.4), (5.5), and (5.7) imply

$$\psi(\xi)\psi(\eta) - \psi(\eta)\psi(\xi) = \psi([\xi, \eta]).$$

This completes the proof. \square

Corollary 5.4. *Let \mathfrak{g} be a finite dimensional quasitriangular Lie bialgebra with r -matrix $r = \sum_i a_i \otimes b_i$ and let $(\mathfrak{q}, \beta, \varphi)$ be a \mathfrak{g} -quasi-Frobenius Lie algebra. Define $\psi : \mathfrak{g}^* \rightarrow \mathfrak{gl}(\mathfrak{q})$, $\xi \mapsto \psi(\xi)$ by*

$$\psi(\xi) := \sum_i \xi(a_i)\varphi(b_i),$$

where $\varphi(b_i) := \varphi_{b_i} : \mathfrak{q} \rightarrow \mathfrak{q}$. Then $(\mathfrak{q}, \beta, \psi)$ is a \mathfrak{g}^ -quasi-Frobenius Lie algebra.*

Proof. Immediate. \square

Proposition 5.5. *Let \mathfrak{g} be a finite dimensional quasitriangular Lie bialgebra with r -matrix $r = \sum_i a_i \otimes b_i$. Let $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, $x \mapsto \varphi(x)$ be a representation of \mathfrak{g} on V . Define $\psi : \mathfrak{g}^* \rightarrow \mathfrak{gl}(V)$, $\xi \mapsto \psi(\xi)$ according to Proposition 5.3. Define $\rho : D(\mathfrak{g}) \rightarrow \mathfrak{gl}(V)$, $a \mapsto \rho(a)$ by*

$$\rho(x + \xi) := \varphi(x) + \psi(\xi), \quad \forall x \in \mathfrak{g}, \xi \in \mathfrak{g}^*. \quad (5.8)$$

Then ρ is a representation of $D(\mathfrak{g})$ on V .

Proof. By Proposition 5.1, it suffices to show that

$$\psi(ad_x^* \xi) - \varphi(ad_x^* x) = \varphi(x)\psi(\xi) - \psi(\xi)\varphi(x). \quad (5.9)$$

We begin by expanding the left side of (5.9). First,

$$\begin{aligned}\psi(ad_x^* \xi) &= \sum_i (ad_x^* \xi)(a_i)\varphi(b_i) \\ &= \sum_i \xi([a_i, x])\varphi(b_i).\end{aligned}\quad (5.10)$$

By Proposition 2.25, $\sum_i a_i \otimes b_i + \sum_i b_i \otimes a_i$ is invariant under the adjoint action of \mathfrak{g} . Hence,

$$\sum_i [a_i, x] \otimes b_i = \sum_i a_i \otimes [x, b_i] + \sum_i [x, b_i] \otimes a_i + \sum_i b_i \otimes [x, a_i]. \quad (5.11)$$

Equations (5.10) and (5.11) now imply

$$\psi(ad_x^* \xi) = \sum_i \xi(a_i) \varphi([x, b_i]) + \sum_i \xi([x, b_i]) \varphi(a_i) + \sum_i \xi(b_i) \varphi([x, a_i]). \quad (5.12)$$

Next, we note that

$$ad_\xi^* x = \sum_i \xi(b_i) [x, a_i] + \sum_i \xi([x, b_i]) a_i. \quad (5.13)$$

From (5.12) and (5.13), we have

$$\psi(ad_x^* \xi) - \varphi(ad_\xi^* x) = \sum_i \xi(a_i) \varphi([x, b_i]). \quad (5.14)$$

For the right side of (5.9), we have

$$\begin{aligned} \varphi(x) \psi(\xi) - \psi(\xi) \varphi(x) &= \sum_i \xi(a_i) \varphi(x) \varphi(b_i) - \sum_i \xi(a_i) \varphi(b_i) \varphi(x) \\ &= \sum_i \xi(a_i) \varphi([x, b_i]) \\ &= \psi(ad_x^* \xi) - \varphi(ad_\xi^* x), \end{aligned} \quad (5.15)$$

where the last equality follows from (5.14). This completes the proof. \square

Theorem 5.6. *Let \mathfrak{g} be a finite dimensional quasitriangular Lie bialgebra. Let $(\mathfrak{q}, \beta, \varphi)$ be any \mathfrak{g} -quasi-Frobenius Lie algebra. Then there exists a representation $\rho : D(\mathfrak{g}) \rightarrow \mathfrak{gl}(\mathfrak{q})$ such that $\rho|_{\mathfrak{g}} = \varphi$ and $(\mathfrak{q}, \beta, \rho)$ is a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra.*

Proof. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be the r-matrix associated to \mathfrak{g} and let $\psi : \mathfrak{g}^* \rightarrow \mathfrak{gl}(\mathfrak{q})$ be the representation of \mathfrak{g}^* on \mathfrak{q} determined by φ and r according to Proposition 5.3. By Corollary 5.4, $(\mathfrak{q}, \beta, \psi)$ is a \mathfrak{g}^* -quasi-Frobenius Lie algebra. Define $\rho : D(\mathfrak{g}) \rightarrow \mathfrak{gl}(\mathfrak{q})$ by

$$\rho(x + \xi) := \varphi(x) + \psi(\xi), \quad \forall x \in \mathfrak{g}, \xi \in \mathfrak{g}^*.$$

By Proposition 5.5, ρ is a representation of $D(\mathfrak{g})$ on \mathfrak{q} . Since $(\mathfrak{q}, \beta, \varphi)$ and $(\mathfrak{q}, \beta, \psi)$ are \mathfrak{g} and \mathfrak{g}^* -quasi-Frobenius Lie algebras and $\rho|_{\mathfrak{g}} = \varphi$ and $\rho|_{\mathfrak{g}^*} = \psi$ (by definition), it follows that $(\mathfrak{q}, \beta, \rho)$ is a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra. \square

Corollary 5.7. *Let \mathfrak{g} be any finite dimensional Lie algebra and let $(\mathfrak{q}, \beta, \varphi)$ be any \mathfrak{g} -quasi-Frobenius Lie algebra. Let $D(\mathfrak{g})$ be the Drinfeld double of the Lie bialgebra (\mathfrak{g}, γ) where $\gamma \equiv 0$. Define $\rho : D(\mathfrak{g}) \rightarrow \mathfrak{gl}(\mathfrak{q})$ by $\rho(x + \xi) = \varphi(x)$ for all $x \in \mathfrak{g}, \xi \in \mathfrak{g}^*$. Then $(\mathfrak{q}, \beta, \rho)$ is a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra.*

Proof. (\mathfrak{g}, γ) is naturally a quasitriangular Lie bialgebra with r-matrix $r \equiv 0 \in \mathfrak{g} \otimes \mathfrak{g}$. Corollary 5.7 now follows as a special case of the proof of Theorem 5.6. \square

We conclude the paper with an example.

Example 5.8. Let (\mathfrak{q}, β) be the 4-dimensional quasi-Frobenius Lie algebra from Example 3.4. For convenience, we recall its structure: \mathfrak{q} has basis $\{e_1, e_2, e_3, e_4\}$ with non-zero commutator relations given by

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = 2e_4, \quad [e_2, e_3] = e_4,$$

and the matrix representation of β with respect to $\{e_1, e_2, e_3, e_4\}$ is

$$(\beta_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}$$

Let $(\mathfrak{g}, \delta r)$ be the 2-dimensional triangular Lie bialgebra from Example 2.27 and 2.29. Once again, we recall the structure for convenience. \mathfrak{g} has basis $\{x, y\}$ with commutator relation $[x, y] = x$ and r-matrix $r = y \wedge x$. Let $\{x^*, y^*\}$ denote the corresponding dual basis. The commutator relations on $D(\mathfrak{g})$ are

$$\begin{aligned} [x, y] &= x, & [x^*, y^*] &= y^*, & [x, x^*] &= -y^*, & [x, y^*] &= 0 \\ [y, x^*] &= x^* + y, & [y, y^*] &= -x \end{aligned}$$

Let $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{q})$ be the linear map defined by

$$\begin{aligned} \varphi_x(e_1) &= 0, & \varphi_x(e_2) &= 0, & \varphi_x(e_3) &= e_2, & \varphi_x(e_4) &= 0 \\ \varphi_y(e_1) &= 0, & \varphi_y(e_2) &= -\frac{1}{2}e_2, & \varphi_y(e_3) &= \frac{1}{2}e_3, & \varphi_y(e_4) &= 0. \end{aligned}$$

Consideration of Example 3.4 (or a direct calculation) shows that $(\mathfrak{q}, \beta, \varphi)$ is a \mathfrak{g} -quasi-Frobenius Lie algebra. By Theorem 5.6, there exists a representation $\rho : D(\mathfrak{g}) \rightarrow \mathfrak{gl}(\mathfrak{q})$ such that $\rho|_{\mathfrak{g}} = \varphi$ and $(\mathfrak{q}, \beta, \rho)$ is a $D(\mathfrak{g})$ -quasi-Frobenius Lie algebra. We now compute ρ explicitly. From the proof of Theorem 5.6, this amounts to computing the representation $\psi : \mathfrak{g}^* \rightarrow \mathfrak{gl}(\mathfrak{q})$ which is determined by φ and $r = y \wedge x$ according to Proposition 5.3:

$$\psi_{x^*} = -\varphi_y, \quad \psi_{y^*} = \varphi_x.$$

ρ is then uniquely defined by $\rho|_{\mathfrak{g}} = \varphi$ and $\rho|_{\mathfrak{g}^*} = \psi$.

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